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NONLINEAR VIBRATIONS OF A NANO SIZED SENSOR WITH FRACTIONAL DAMPING

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A nano - sized oscillator with fractional damping and a nonlinearity due to the Casimir force is presented. It is used the Fourier - Volterra series method for the study of the nonlinear mode. The solution is established also using a variational integral method of J. H. He, based on Lagrange multiplier method.

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1. Introduction

For many years the Casimir effect was a theoretical curiosity. Experimental physicists have realized that the Casimir force affects the micro-machined devices.

The Casimir effect [1] represents a quantum electrodynamics effect [2] which consists in polarization of two perfectly conducting bodies. The Casimir force which appear between the two conductors acquire significant values when separation between the conducting surfaces is reduced to less than 100 *nm*. This effect was established also between two dielectric media placed in the field [4].

The Casimir effect becomes important only in our days due to its manifestations in the micro- and nano-structures. This effect presents a series of applications at the nano/microelectromechanical systems (NEMS/MEMS) used in mechanical sensors [3,4] and at the future quantum computing systems. It is important to tell that the motion of this class of sensors can be described equally in terms classical mechanics or in terms of quantum mechanics.

The Casimir effect appears also in conductors with finite conductance and in conductors with rugosity [5].

On the other hand it is found that in the NEMS/MEMS structures the most adequate kind of damping process is the factional damping [6].

The mathematical modeling of systems with nonlinearity and fractional damping is difficult, being used currently numerical procedures.

The aim of this paper is to illustrate the possibility of obtaining good approximate solutions of motion equations for a NEMS/MEMS system with nonlinearity and factional damping using the He variational integral method [7,8] and the Volterra series method [9,10].

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2. Mechanical model

There was established that the Casimir force between two perfectly conducting plates without rugosity, is an attractive force given by [1,2]:

$$F_c = \frac{\pi^2 \hbar c A}{240 z^4} \tag{1}$$

and between a sphere and a conducting plate, perfectly conducting and smooth surfaces is given by:

$$F_c = \frac{\pi^3 \hbar c R}{360 z^3} \tag{2}$$

where $\hbar = h/2\pi$ is the Planck constant, c – is the speed of the light, A – the area of the plates, R – radius of the sphere and z is the distance between the two plates, respectively between the sphere and the plate.

In practice, it is preferable the configuration plate – sphere in order to avoid the alignment problems of the parallel plates.

The system is presented in the figure 1.

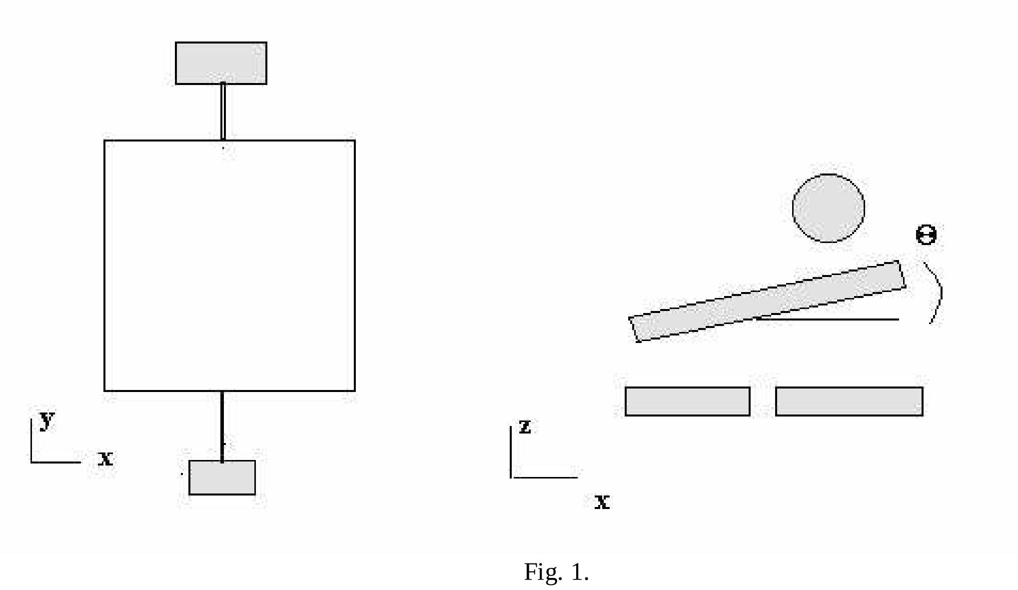

Fig. 1.

Our model comsistes of a micro sized cantilever subjected to an elstic force, a Casimir force and a fractional damping force [2]. The motion equation of the system is describet by:

$$m\frac{d^2}{dt^2}x + \lambda\frac{d^f}{dt^f}x + kx - \frac{c}{(d-x)^4} = F(t) \tag{3}$$

where the second term is due to the fractional friction force, the third to the elastic force, and the forth to the Casimir force and *F(t)* an external excitation force. The fractional damping in expressed by a fractional derivative, f being a fractional number, and λ - a damping constant. A typical value for f can be taken 0.5. The Casimir force correspond to the case of conducting plate spaced at d in the equilibrium position from a conducting surface. m represents the mass of the conducting plate and k the equivalent elastic constant of the suspension. The constant c results from (1):

$$c = \frac{\pi^2 \hbar c A}{240}$$

The d^f/dt^f fractional derivative represents a linear operator intuitively defined [11] in terms of Fourier analysis. If we denote by $X(\omega)$, the Fourier transform of $x(t)$:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\exp(-i\omega t)dt,$$

the $d^f x/dt^f$ fractional derivative of x(t) will be:

$$\frac{d^f}{dt^f}x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(i\omega)^f\, X(\omega)\exp(i\omega t)dt, \tag{4}$$

We consider the case of a harmonic excitation with angular frequency w will denote by w_n the natural angular frequency of the system in absence of the Casimir force and damping:

$$F(t) = F_0\cos\omega t, \qquad \omega_n = \sqrt{\frac{k}{m}}, \tag{5}$$

where F_0 is the amplitude of the force.

The equation (3) becomes:

$$\frac{d^2}{dt^2}x + \frac{\lambda}{m}\frac{d^f}{dt^f}x + \omega_n{}^2 x - \frac{1}{m}\frac{c}{(d-x)^4} = \frac{1}{m}F_0\cos\omega t \tag{6}$$

A similar equation can be obtained for the case of a spherical conductor sustained by an elastic element with h constant (a cantilever). In this case in the fouth term of (6) the power of the denominator will be 3 and c:

$$c = \frac{\pi^3 \hbar c R}{360}.$$

The nonlinear term can be expressed in the case of small oscillation in series, resulting that (5) can be written:

$$\frac{d^2}{dt^2}x + \frac{\lambda}{m}\frac{d^f}{dt^f}x + \omega_n{}^2 x - \frac{c}{md}\left(1 + 4\frac{\mathrm{x}}{\mathrm{d}} + 10\frac{\mathrm{x}^2}{\mathrm{d}^2} + 20\frac{\mathrm{x}}{\mathrm{d}^3}{}^3 + 35\frac{\mathrm{x}^4}{\mathrm{d}^4}\right) = \frac{1}{m}F_0\cos\omega t \tag{7}$$

3. Variational iteration method

In order to find an analytical solution of the Eq.(7) we will use a powerful variational iteration method created by J. H. He [5], similar with Lagrange multiplier method.

This method gives the possibility to give convenient approximate solutions to all kinds of non-linear equations. We limit ourselves to the application of the powerful method for the case of a nonlinear system in the type:

$$Lx + Nx = g(t)\,, \tag{8}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a know function.

The variational iteration method gives the possibility to write the solution of the equation (8) with the aid of the correction functional:

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\tau)\left[Lx_n(\tau) + N\tilde{x}_n(\tau) - \mathrm{g}(\tau)\right]d\tau, \tag{9}$$

where x_n is an initial approximations with possible unknowns, λ is a Lagrange multiplier and $\tilde{x}_n$ represents a term with a restricted variation $\left(\delta\tilde{x}_n = 0\right)$.

The λ multiplier can be found from the stationary condition of the correction functional $\delta x_{n+1}=0$.

We will use this method in order to obtain the solution of the constitutive equation (6), taking:

$$L = \frac{d^2}{dt^2}x + \omega_n{}^2 x,$$

$$N = \frac{\lambda}{m}\frac{d^f}{dt^f}x - \frac{1}{m}\frac{c}{(d-x)^4}\,, \tag{10}$$

$$g(t) = \frac{1}{m}F_0 \cos\omega t$$

The correction functional will be:

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\tau)\left[\frac{d^2}{dt^2}x_n + \omega_n{}^2 x_n + N\tilde{x}_n(\tau) - g(\tau)\right]d\tau. \tag{11}$$

Imposing the stationary condition ($\delta x_{n+1}=0$) to the correction functional results in:

$$1 - \lambda'(\tau)\,|_{r=t} = 0\,, \tag{12}$$

$$\lambda''(\tau) + \omega_n\lambda(\tau) = 0, \tag{13}$$

$$\lambda(\tau)\,|_{\tau=t} = 0, \tag{14}$$

from which the Lagrange multiplier can be identified as

$$\lambda(\tau) = \frac{1}{\omega_n}\sin\omega_n(\tau - t)\,. \tag{15}$$

We will consider an initial approximation of the solution of the equation

$$\frac{d^2}{dt^2}x_0 + \omega_n{}^2 x_0 = 0,$$

containing the integration constants C and D:

$$x_0 = C\sin \omega_n t + D\cos \omega_n t. \tag{16}$$

Using (10), (11), (15) and (16) it results a first iteration of the solution:

$$\begin{aligned}
x_1 = {} & C\sin(\omega_n t) + \mathrm{D}\cos(\omega_n t) + \Big(-\lambda\,\omega_n{}^f C\,\omega_n{}^2 \cos(\omega_n t)\cos\left(\frac{1}{2}\pi f\right) \\
& + \lambda\,\omega_n{}^f C\,\omega_n{}^2 \sin(\omega_n t)\sin\left(\frac{1}{2}\pi f\right) + \lambda\,\omega_n{}^f C\,\omega^2 \cos(\omega_n t)\cos\left(\frac{1}{2}\pi f\right) \\
& - \lambda\,\omega_n{}^f C\,\omega^2 \sin(\omega_n t)\sin\left(\frac{1}{2}\pi f\right) - \lambda\,\omega_n{}^f \mathrm{D}\,\omega_n{}^2 \sin(\omega_n t)\cos\left(\frac{1}{2}\pi f\right) \\
& - \lambda\,\omega_n{}^f \mathrm{D}\,\omega_n{}^2 \cos(\omega_n t)\sin\left(\frac{1}{2}\pi f\right) + F_0\cos(t\,\omega)\,\omega_n{}^2 \\
& + \lambda\,\omega_n{}^f \mathrm{D}\,\omega^2 \sin(\omega_n t)\cos\left(\frac{1}{2}\pi f\right) + \lambda\,\omega_n{}^f \mathrm{D}\,\omega^2 \cos(\omega_n t)\sin\left(\frac{1}{2}\pi f\right) \\
& - \lambda\,\omega_n{}^f C\,\omega^2 \cos(\omega_n t)^2\cos\left(\frac{1}{2}\pi f\right) + \cos(\omega_n t)\,\lambda\,\omega_n{}^f C\,\omega^2 \sin(\omega_n t)\sin\left(\frac{1}{2}\pi f\right) \\
& - F_0\cos(\omega_n t)\,\omega_n{}^2 + \cos(\omega_n t)\,\lambda\,\omega_n{}^f \mathrm{D}\,\omega_n{}^2 \sin(\omega_n t)\cos\left(\frac{1}{2}\pi f\right) \\
& + \lambda\,\omega_n{}^f \mathrm{D}\,\omega_n{}^2 \cos(\omega_n t)^2\sin\left(\frac{1}{2}\pi f\right) - \cos(\omega_n t)\,\lambda\,\omega_n{}^f \mathrm{D}\,\omega^2 \sin(\omega_n t)\cos\left(\frac{1}{2}\pi f\right) \\
& - \lambda\,\omega_n{}^f \mathrm{D}\,\omega^2 \cos(\omega_n t)^2\sin\left(\frac{1}{2}\pi f\right) + \lambda\,\omega_n{}^f C\,\omega_n{}^2 \cos(\omega_n t)^2\cos\left(\frac{1}{2}\pi f\right) \\
& - \cos(\omega_n t)\,\lambda\,\omega_n{}^f C\,\omega_n{}^2 \sin(\omega_n t)\sin\left(\frac{1}{2}\pi f\right)\Big) \Big/ \left(m\,\omega_n{}^2(\omega_n + \omega)(\omega_n - \omega)\right) + w
\end{aligned} \tag{17}$$

where:

$$\begin{aligned}
w = {} & \frac{1}{24}\big(120\,c\,\mathrm{D}^2 d^2 + 120\,c\,C^2 d^2 - 180\,\mathrm{D}^3 \sin(\omega_n t)\,\omega_n\,m\,t + 48\,\mathrm{D}\sin(\omega_n t)\,\omega_n\,c\,d^3 t \\
& + 180\,C\cos(\omega_n t)\,\omega_n\,\mathrm{D}^2 m\,t - 48\,C\cos(\omega_n t)\,\omega_n\,c\,d^3 t + 180\,C^3\cos(\omega_n t)\,\omega_n\,m\,t \\
& - 180\,\mathrm{D}\sin(\omega_n t)\,\omega_n\,C^2 m\,t - 80\,c\,C\,\mathrm{D}\sin(2\,\omega_n t)\,d^2 + 48\,C\sin(\omega_n t)\,c\,d^3 \\
& + 45\,C\sin(3\,\omega_n t)\,m\,\mathrm{D}^2 - 315\,C\sin(\omega_n t)\,\mathrm{D}^2 m - 15\,\mathrm{D}^3\cos(\omega_n t)\,m \\
& + 15\,\mathrm{D}^3\cos(3\,\omega_n t)\,m - 135\,C^3\sin(\omega_n t)\,m - 45\,\mathrm{D}\cos(3\,\omega_n t)\,m\,C^2 \\
& + 40\,c\cos(2\,\omega_n t)\,d^2 C^2 - 40\,c\cos(2\,\omega_n t)\,d^2\mathrm{D}^2 + 45\,\mathrm{D}\cos(\omega_n t)\,C^2 m \\
& + 160\,c\,C\,\mathrm{D}\sin(\omega_n t)\,d^2 - 80\,c\cos(\omega_n t)\,d^2\mathrm{D}^2 - 160\,c\cos(\omega_n t)\,d^2 C^2 \\
& - 24\,c\cos(\omega_n t)\,d^4 + 24\,d^4 c - 15\,C^3\sin(3\,\omega_n t)\,m\big) \Big/ \left(\omega_n{}^2 m\,d^3\right)
\end{aligned} \tag{18}$$

These calculations was obtained on computer with the aid of the symbolic calculation package Maple 7. It is possible to obtain a second iteration.

In the calculations we used the fractional differentiation rules [11]:

$$\frac{d^f}{dt^f}\sin(at) = a^f \sin(at+\frac{\pi f}{2}), \qquad \frac{d^f}{dt^f}\cos(at) = a^f \cos(at+\frac{\pi f}{2}). \qquad (19)$$

It results that the variational iteration method gives analytic accurate solutions. In the linear case this method gives after one iteration the exact solution for the oscillator with fractional damping. It is possible to use this method to identify the nonlinear models from experimental results [12].

3. Volterra series method

There are methods expressing the response of the nonlinear systems in terms of generalized transfer functions. For a nonlinear system described by equations, the solution *x(t)* can be expressed in terms of Volterra series as [9,10]:

$$x(t) = \int_{-\infty}^{\infty} h_1(\tau_1)u(\tau_1 - t)d\tau_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)u(\tau_1 - t)u(\tau_2 - t)d\tau_1 d\tau_2 + \ldots$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,\ldots\tau_n)u(t-\tau_1)u(t-\tau_2)\ldots u(t-\tau_n)d\tau_1 d\tau_2\ldots d\tau_n + \ldots \qquad (20)$$

where $h_n(\tau_1, \tau_2, \ldots, \tau_n)$ is the n^{th} order ($n = 1, 2, 3, \ldots$) transfer function, and u represents the input excitation. In our case $u(t) = (F_0/m)\exp(i\omega t)$. The form of this kind of nonlinear transfer function depend on the due form of excitation function $u(t)$.

The n-dimensional Fourier transform of the n^{th} order transfer function is:

$$H_n(\omega_1, \omega_2, \ldots, \omega_n) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,\ldots\tau_n)\exp[-i(\omega_1\tau_1+\omega_2\tau_2+\ldots\omega_n\tau_n)]d\tau_1 d\tau_2\ldots d\tau_n$$

and by inverse transform results:

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} H_n(\omega_1,\omega_2,\ldots\omega_n)\exp[i(\omega_1\tau_1+\omega_2\tau_2+\ldots\omega_n\tau_n)]d\omega_1 d\omega_2\ldots d\omega_n$$

We consider in the following that the excitation is of harmonic type $u = A\exp(i\,\omega t)$, where A is the amplitude of the external perturbation and ω is the angular frequency.

The Volterra series method gives the possibility to obtain solutions for nonlinear differential equations with the form:

$$Lx + \sum_{k=2}^{n} a_k x^k = A\exp(i\omega t),$$

where the second term contain a polynomial nonlinearity, and L is a linear operator which contains the derivatives with respect to time, and constant quotients. In our case

$$L = \frac{d^2}{dt^2} + \frac{\lambda}{m}\frac{d^f}{dt^f} + (\omega_n{}^2 - \frac{4c}{md^2}),$$

By introducing eq. (20) in eq. (7) one may calculate different orders of the transfer function for harmonic excitation. In our case the response will have the form [9,10]:

$$x(t) = \sum_{n=1}^{\infty} (\frac{F_0}{m})^n H_n(\omega_1, \omega_2, ..., \omega_n) \exp(in\omega t) = \sum_{n=1}^{\infty} \xi_n \exp(in\omega t) \tag{21}$$

where $\omega_1 = ... = \omega_n = \omega$. This solution represents a superposition of harmonics. The amplitudes ξ_n can be found from the motion equation (7).

Using the notation $L(\omega) = -\omega^2 + (\lambda/m)(i\omega)^f + (\omega_n{}^2 - 4c/md^2)$ the following results are obtained:

$$\begin{aligned} \xi_1 &= \frac{1}{L(\omega)} \\ \xi_2 &= -\frac{1}{L(2\omega)} 2a_2 \xi_1^2 \\ \xi_3 &= -\frac{1}{L(3\omega)} (6a_2 \xi_1 \xi_2 + 6\xi_1{}^2) \\ &\dots \\ \xi_n &= -\frac{1}{L(n\omega)} \sum_{k=2}^{n} \alpha_k H(\omega, \omega, ..., \omega) \end{aligned} \tag{22}$$

where $\alpha_2 = -10c/md^3$, $\alpha_3 = -20c/md^4$ and $\alpha_4 = -35c/md^4$.

In the last equation $H_n^{(k)}$ (where $0 \le k \le n$) represents the modified transfer function which correspond to the x^k response [8,9]:

$$H^{(k)}{}_n(\omega_1, \omega_2, ..., \omega_n) =$$

$$= k! \sum_{(\nu,k,n)} \sum_N{}' H_{\nu_1}(\omega_1, \omega_2, \ldots, \omega_{\nu_1}) H_{\nu_2}(\omega_{\nu_1+1}, \omega_{\nu_1+2}, \ldots, \omega_{\nu_2}) \ldots H_{\nu_k}(\omega_\mu, \omega_2, \ldots, \omega_{\nu_n})$$

where $\omega_1 = ... = \omega_n = \omega$ and the notations:

$$\nu = \nu_1 + \nu_2 + ... + \nu_{k-1} + 1 = n - \nu_k + 1,$$

were used; (ν, k, n) indicates that the summation will be made for all integer values ν_j so that

$$\nu_1 + \nu_2 + ... + \nu_k = n, \qquad 1 \le \nu_1 \le \nu_2 \le ... \le \nu_k.$$

Σ'_N indicates that the summation will be repeated N times, corresponding to all non-identical products which can be obtained by permutation of the index j of ω_j. The number of terms is:

$$N_0 = \frac{n!}{\nu_1! \nu_2! ... \nu_k! r_1! r_2! ... r_k!}$$

where r_1 is the number of the first number of index equally, and so on.

It is important to note that the Volterra series method gives the possibility to obtain the parameters of the nonlinear model from the harmonic components of the response [13].

4. Conclusions

In the paper we consider a MEMS/NEMS nonlinear oscillator with fractional damping and nonlinearity due to the Casimir force. These models of nonlinear oscillators is of interest due its applcations in the field of micr/nano sensor structures.

The aim of the paper was to demonstrate that it is possible to obtain analytic solution with accurate precision in the case of nonlinearities and fractional damping.

An accurate solution of this nonlinear oscillator was obtained without difficulty using the variational iteration method of He. In the linear case this method gives, after one iteration, the exact solution for the oscillator with fractional damping.

A solution of the nonlinear oscillator corresponding to a harmonic excitation $(F_0/m)\exp(i\omega t)$ was obtained in terms of harmonic components. The Volterra series method gives the possibility to obtain the parameters of the nonlinear model from the harmonic components of the response.

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