The inverse problem in scattering theory of optical fields

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A general model was built for spatial solitons in photorefractive crystals using the inverse problem in the scattering theory. The inverse problem in the scattering theory is defined knowing the spectral data that characterize the scattering. We present a formalism regarding the use of the inverse method in solving the nonlinear differential equations. Envelope singular analytical solutions (solitons) and asymptotically solutions of the wave equation for integral equation (of SBS type) were obtained. The results are in good agreement with the results obtained in other papers.

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1. Introduction

In this paper we build up a model for spatial solitons in photorefractive crystals process using the inverse problem in scattering theory.


Years’70 are characterized by the effort made for solving of nonlinear differential equations using the inverse problem and for singular envelope solutions (solitons) description.


A formalism of the inverse method in solving nonlinear differential equations (of SBS type) and for the interpretation optical spatial solitons in photorefractive crystals will be presented in this paper.

2. Mathematical

It is considered tools the wave equation in the form [24]:

$$\Delta \Phi + k^2 \Phi - V(r)\Phi = 0.$$  (1)

Choosing $V(r)$ with spherical symmetry we use the development:

$$\Phi(\mathbf{r}) = \frac{\phi(r)}{r^{\lambda}} Y_m^l(\theta, \varphi).$$  (2)

The equation for radial component yields [24]:

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d \phi}{dr} + k^2 \phi - \frac{\lambda^2}{r^2} \phi - V(r) \phi = 0$$  (3)

where $\lambda = l + 1/2$. Where $i = 0, 1, 2, 3, \ldots; j = 0, 1, 2, 3, 4, 5, 6, 7, \ldots$

The function $\phi = \phi(r) \rightarrow \phi(r | k, \lambda)$ is a regular solution in $r = 0$. We impose for $V(r)$ the conditions:

$$\int_0^r |V(r)| dr < \infty \text{ (finite value)}$$  (4)

$$\int_0^\infty |V(r)| dr < \infty \text{ (finite value)}$$  (5)
If the conditions considered (4) and (5) out it results:

$$\lim_{r \to \infty} \varphi(r | k, \lambda) = \sqrt{\frac{2}{\pi r}} A(k, \lambda) \cdot \sin \left[ kr - \frac{\pi}{2} \left( \lambda - \frac{1}{2} \right) + \delta(k, \lambda) \right]$$

(6)

In the asymptotical expression of $\varphi(r)$ defined in (6) we use notations for:

$A(k, \lambda) \to$ the wave scattering amplitude 

$\delta(k, \lambda) \to$ the wave scattering phase.

With these data, one defines the inverse problem in scattering theory: $V(r)$ will be determined knowing the spectral data which characterize the scattering ($A(k, \lambda)$, $\delta(k, \lambda)$).

It will be analyzed later the case in which $k=1$. So, the spectral data take the form:

$$A(l, \lambda) = A(\lambda) \mid_{l_0 = l_1 = \frac{1}{2}, l = 0, 1, 2, ...} \quad \delta(l, \lambda) = \delta(\lambda) \mid_{l_0 = l_1 = \frac{1}{2}, l = 0, 1, 2, ...}$$

(7)

The wave equation (3) can be written in integral form (as proposed by Regge [5]):

$$\varphi(r, \lambda) = I_{\lambda}(r) + \int_{0}^{r} k(r, \rho) \cdot I_{\lambda}(\rho) \frac{d\rho}{\rho}$$

(8)

$I_{\lambda}(r)$ → are the first species Bessel functions modified of ($\lambda$) order.

Using Regge integral solution (8) and wave equation (3), we build up the operators:

$$\hat{D}(r) = r^2 \left[ \frac{d^2}{dr^2} + \frac{1}{r} \cdot \frac{d}{dr} + 1 - V(r) \right]$$

$$\hat{D}_0(\rho) = \rho^2 \left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \cdot \frac{d}{d\rho} + 1 \right]$$

$$\hat{\Delta}(r) = r^2 \left[ \frac{d^2}{dr^2} + \frac{1}{r} \cdot \frac{d}{dr} \right]$$

$$\hat{B}_\lambda = r^2 \left[ \frac{d^2}{dr^2} + \frac{1}{r} \cdot \frac{d}{dr} + 1 - \frac{\lambda^2}{r^2} \right] \cdot \hat{B}_\lambda \cdot I_{\lambda}(r) = 0$$

The wave equation as an equation with eigenvalues is written on the form:

$$\hat{D}(r) \varphi(r, \lambda) = \lambda^2 \varphi(r, \lambda)$$

(10)

In the following will be presented the method of R. Newton of solving the inverse problem. We attach to the integral equation (8) and to the operators system (9) the equation with eigenvalues on the form [25]:

$$\hat{D}(r) \cdot K(r, \rho) = \hat{D}_0(\rho) \cdot K(r, \rho)$$

$$V(r) = \frac{2}{r^2} \int_{0}^{r} \left[ \varphi(r, \lambda) + \frac{1}{2} \frac{d}{dr} \varphi(r, \lambda) + \frac{1}{2} \frac{d}{dr} \varphi(r, \lambda) \right]$$

(12)

and thus we obtain an integral equation similar to that in (8), but for the integral nucleus $K(r, \rho)$:

$$K(r, \rho) = F(r, \rho) + \int_{0}^{r} K(r, \zeta) \cdot F(\zeta, \rho) \frac{d\zeta}{\rho} = \left[ \hat{D}(r) K(r, \rho) = \hat{D}_0(\rho) K(r, \rho) \right]$$

$$\varphi(r, \lambda) = I_{\lambda}(r) + \int_{0}^{r} K(r, \rho) \cdot I_{\lambda}(\rho) \frac{d\rho}{\rho} = \left[ \hat{D}(r) \varphi(r, \lambda) = \lambda^2 \varphi(r, \lambda) \right]$$

(15)

By introducing the development (14) in integral equations (15), we obtain the integral nucleus expression $K(r, \rho)$ on the form [26]:

$$K(r, \rho) = \sum_{l=0}^{\infty} C_{l+\frac{1}{2}} \cdot I_{l+\frac{1}{2}}(\rho) \cdot \varphi(r, l + 1/2)$$

(16)

The potential function $V(r)$ of (12) takes the form [26, 27]:

$$V(r) = \frac{2}{r^2} \sum_{l=0}^{\infty} C_{l+\frac{1}{2}} \left[ \varphi(r, l + \frac{1}{2}) \frac{d}{dr} \varphi(r, l + \frac{1}{2}) + I_{l+\frac{1}{2}}(\rho) \frac{d\varphi(r, l + 1/2)}{dr} \right]$$

(17)

where:

$$\varphi(r, l + \frac{1}{2}) = I_{l+\frac{1}{2}}(r) + \sum_{l=0}^{\infty} C_{l+\frac{1}{2}} \cdot \varphi(r, l + \frac{1}{2})$$

(17')

where:

$$L_{\varphi}(r) = \int_{0}^{r} I_{l+\frac{1}{2}}(\rho) \cdot I_{l+\frac{1}{2}}(\rho) \frac{d\rho}{\rho}$$

(18)

The algebraic equations system (17') is solved in comparison with unknown $C_{l+\frac{1}{2}}$, which later are introduced in (17) for the determination of the potential function $V(r)$.
One calculates the dependence of the coefficients $C_{\frac{r_1}{2}}$ as a function of spectral data (phase variation: $\delta(1, l + 1/2)$)

The algebraic system (17') is multiplied by $\sqrt{\frac{\pi r}{2}}$ and one gets the system:
\[
\sqrt{\frac{\pi r}{2}} \phi(r, l + 1/2) = \sqrt{\frac{\pi r}{2}} l_{\frac{r_1}{2}} + \sum_{r=0}^{r_1} C_{\frac{r_1}{2}} \sqrt{\frac{\pi r}{2}} \phi(r, l' + 1/2), L_0(r)
\]

The equation system (18) cross to the limit $r \to \infty$; thus results:
\[
\lim_{r \to \infty} \sqrt{\frac{\pi r}{2}} \phi(r, l + 1/2) = A \left(1, l + 1/2\right) \sin \left[r - \frac{\pi l}{2} + \delta \left(1, l + 1/2\right)\right],
\]

Under the circumstances, the system becomes:
\[
A \left(1, l + 1/2\right) \sin \left[r - \frac{\pi l}{2} + \delta \left(1, l + 1/2\right)\right] = \lim_{r \to \infty} \sqrt{\frac{\pi r}{2}} I_{\frac{r_1}{2}}(r) + \sum_{r=0}^{r_1} C_{\frac{r_1}{2}} A \left(1, l' + 1/2\right) \sin \left[r - \frac{\pi l'}{2} + \delta \left(1, l' + 1/2\right)\right], L_0(r)
\]

We use asymptotic solutions for Bessel functions:
\[
\lim_{r \to \infty} I_{\frac{r_1}{2}}(r) = \frac{2}{\pi r} \cdot \sin \left(r - \frac{\pi l}{2}\right)
\]

Thus, results:
\[
\lim_{r \to \infty} L_0(r) = \frac{2}{\pi} \cdot \sin \left[\frac{\pi}{2} (l - l')\right] (l + l' + 1)(l - l')
\]

Introducing these relations in algebraic equation system (20) we obtain an algebraic unknown equation

system
\[
\left[\begin{array}{cccc}
C_{\frac{r_1}{2}}
\end{array}\right]^{[28-30]}:
\]

\[
A \left(1, l + 1/2\right) e^{-i\left(l - l_1\right)} + \sum_{r=0}^{r_1} C_{\frac{r_1}{2}} A \left(1, l' + 1/2\right)e^{-i\left(l - l'_1\right)}
\]

We observe that the asymptotic equation system (23) is no longer dependent on $r$, it only ensures the coefficients dependence $C_{\frac{r_1}{2}}$ function of spectral data

\[
\left\{A \left(1, l + 1/2\right); \delta \left(1, l + 1/2\right)\right\}
\]

therefore implicit and the structural dependence of the $V(r)$ potential function of spectral data.

We perform the spectral analysis of the following operators $[31,32]$:

\[
\hat{\Delta}_0(r) = r \cdot \frac{d}{dr} \left[\frac{d}{dr} + \frac{r^2}{r^2} \right] \rightarrow \text{(nondisturbed operator)}
\]

\[
\hat{D}_0(r) = r \cdot \frac{d}{dr} \left[\frac{d}{dr} + r^2 \right] \rightarrow \text{(nondisturbed operator)}
\]

\[
\hat{D}(r) = r \cdot \frac{d}{dr} \left[\frac{d}{dr} + r^2 - r^2 V(r) \right] \rightarrow \text{(disturbed operator)}
\]

We use Fourier transforms in the form:
\[
F(\tau) = \int_{-\infty}^{\infty} f(x) e^{-i\tau x} dx
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\tau) e^{i\tau x} d\tau
\]

One makes the substitution $e^t = r$ and then the Fourier transforms takes the form:
\[
F(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(r) e^{-i\tau \rho} r \frac{dr}{r^3} \bigg|_{\rho = 0(r)}
\]

\[
g(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\tau) e^{-i\tau \rho} r^3 d\tau \bigg|_{\rho = 0(r)}
\]

The spectra associated to the operators (24) (i.e. the solutions) has the form:

\[
\hat{\Delta}_0 \rightarrow \hat{\Delta}_0(r) \varphi(r) = r^2 \varphi(r); \ \varphi_0 = \frac{1}{2^2} \Gamma(1 + \lambda);
\]

\[
\hat{D}_0 \rightarrow \hat{D}_0(r) I_1(r) = \lambda \varphi(r); \ \hat{D} \rightarrow \hat{D}(r) \varphi(r)
\]

where $I_1(r)$ are I species and I order – modified Bessel functions.

We use the crossing operators $\{\hat{X}_{\Delta_0, \Delta_0}, \hat{X}_{\Delta_0, \Delta_0}, \hat{X}_{\Delta_0, \Delta_0}\}$ defined as:

\[
\hat{X}_{\Delta_0, \Delta_0} = I \int_{0}^{r} A_{\Delta_0, \Delta_0}(r, \rho) \cdot \frac{d\rho}{\rho}
\]

\[
\hat{X}_{\Delta_0, \Delta_0} = I \int_{0}^{r} A_{\Delta_0, \Delta_0}(r, \rho) \cdot \frac{d\rho}{\rho}
\]

\[
\hat{X}_{\Delta_0, \Delta_0} = I \int_{0}^{r} A_{\Delta_0, \Delta_0}(r, \rho) \cdot \frac{d\rho}{\rho}
\]
The application method of the crossing operators [33-36]:
\[ \dot{X}_{\lambda n,0} \varphi(r,\lambda) = \varphi_\circ(r,\lambda) + \int_0^1 A_{\lambda n,D}(r,\rho) \varphi_0(\rho,\lambda) \frac{d\rho}{\rho} \]
\[ \dot{X}_{\lambda n,0}^0 = \dot{X}_{\lambda n,0} \varphi_0(r,\lambda) = \varphi(r,\lambda) - \int_0^1 \hat{A}_{\lambda n,D}(r,\rho) \varphi_0(\rho,\lambda) \frac{d\rho}{\rho} \]
\[ (29) \]
where:
\[ \varphi_0(r,\lambda) = \frac{r^k}{2^k \Gamma(1+\lambda)} \]
\[ (30) \]

Using the representation (28) and (29), we build up the fundamental integral equation for solving the inverse problem (the integral Gelfand-Levitan-Marcenko equation; GLM).

We use the integral equation:
\[ \varphi(r,\lambda) = \varphi_0(r,\lambda) + \int_0^1 A_{\lambda n,D}(r,z) \varphi_0(z,\lambda) \frac{dz}{z} \]
\[ (31) \]

Given the case of discrete spectrum \( \lambda^2 > 0 \); \( \lambda \in \mathbb{R} \);
\[ \lambda \to \lambda_n \text{ for } n \to 0 \text{ from 1 to infinite.} \]
\[ \sum_{n=1}^{\infty} \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi(r,\lambda_n) = \sum_{n=1}^{\infty} \varphi_0(\rho,\lambda_n) \varphi_0(r,\lambda_n) + \int_0^r A_{\lambda n,D}(r,z) \left[ \sum_{n=1}^{\infty} \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi_0(z,\lambda_n) \right] \frac{dz}{z} \]
\[ (33) \]

We divide the relation (32) by \( a_n^2 \) and we count after \( n \) from 1 to infinite.
\[ \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi(r,\lambda_n) = \sum_{n=1}^{\infty} \varphi_0(\rho,\lambda_n) \varphi_0(r,\lambda_n) + \int_0^r A_{\lambda n,D}(r,z) \left[ \sum_{n=1}^{\infty} \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi_0(z,\lambda_n) \right] \frac{dz}{z} \]
\[ \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi(r,\lambda_n) = \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi_0(r,\lambda_n) + \int_0^r A_{\lambda n,D}(r,z) \left[ \sum_{n=1}^{\infty} \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi_0(z,\lambda_n) \right] \frac{dz}{z} \]
\[ (33) \]

Given the case of negative spectrum (continuum spectrum): \( \lambda^2 < 0 \); \( \lambda \in \mathbb{R} \).
\[ \lambda \to \lambda_n \text{ for } n \to 0 \text{ from 1 to infinite.} \]
\[ \sum_{n=1}^{\infty} \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi(r,\lambda_n) = \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi_0(r,\lambda_n) + \int_0^r A_{\lambda n,D}(r,z) \left[ \sum_{n=1}^{\infty} \frac{1}{a_n^2} \varphi_0(\rho,\lambda_n) \varphi_0(z,\lambda_n) \right] \frac{dz}{z} \]
\[ (33) \]

We start in the case of discrete spectrum with the crossing integral equation written in the form:
\[ \varphi(r,i\tau) = \varphi_0(r,i\tau) + \int_0^r A_{\lambda n,D}(r,z) \varphi_0(z,i\tau) \frac{dz}{z} \]
\[ (34) \]

We multiply (34) with the integral (equivalent Fourier transform):
\[ \left[ \int_{-\infty}^{\infty} \varphi_0(\rho,-i\tau) \frac{\tau \cdot d\tau}{\sin(\pi \tau)} \right] \]
\[ \text{The equation (34) becomes:} \]
\[ \varphi(r,i\tau) = \varphi_0(r,i\tau) + \int_0^r A_{\lambda n,D}(r,z) \varphi_0(z,i\tau) \frac{dz}{z} \]
\[ (34) \]

We use several integral representations of the form:
\[ \frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\tau \cdot d\tau}{\sin(\pi \tau)} \cdot \rho \cdot r \cdot \phi(r,i\tau) \frac{\tau \cdot d\tau}{\sin(\pi \tau)} = \frac{1}{2 \pi} \int_{-\infty}^{\infty} \rho \cdot r \cdot \phi(r,i\tau) \frac{\tau \cdot d\tau}{\sin(\pi \tau)} \]
\[ (35) \]

We multiply (34) with the integral (equivalent Fourier transform):
\[ \int_{-\infty}^{\infty} \varphi_0(\rho,-i\tau) \frac{\tau \cdot d\tau}{\sin(\pi \tau)} \]
\[ \text{The equation (34) becomes:} \]
\[ \varphi(r,i\tau) = \varphi_0(r,i\tau) + \int_0^r A_{\lambda n,D}(r,z) \varphi_0(z,i\tau) \frac{dz}{z} \]
\[ (34) \]

The function \( F_{\lambda n,D}(r,\rho) \) becomes \( f_{\lambda n,D}(r \cdot \rho) \) (function of product \( r \cdot \rho \)) and yields:
\[ f_{\lambda n,D}(r \cdot \rho) = \sum_{n=1}^{\infty} a_n^2 \cdot \Gamma(1+\lambda_n) \left[ \frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\tau \cdot d\tau}{\sin(\pi \tau)} \right] \frac{\tau \cdot d\tau}{\sin(\pi \tau)} \]
\[ (41) \]
The function \( f_{\lambda,\rho}(r \cdot \rho) \) is named also the function Regge-Loeffel [6]. The integral equation (GLM) has the form:

\[
A_{\lambda,\rho}(r, \rho) + f_{\lambda,\rho}(r \cdot \rho) + \int_0^z A_{\lambda,\rho}(z, r) \cdot f_{\lambda,\rho}(z \cdot \rho) \, dz = 0
\]

(42)

We choose a crossing operator \( \hat{X}_{\lambda,\rho} \) with the form:

\[
\hat{X}_{\lambda,\rho} \varphi_0(r, \lambda) = I_\lambda(r)
\]

(43)

The associated integral equation has the form:

\[
I_\lambda(r) = \varphi_0(r, \lambda) + \int_0^r A_{\lambda,\rho}(r, z) \cdot \varphi_0(z, \lambda) \, dz.
\]

(44)

We build up the spectrum of the operator \( \hat{\Delta}_0 \), based on the eigenvalues equation:

\[
\hat{\Delta}_0(r) \varphi_0(r, \lambda) = \lambda^2 \varphi_0(r, \lambda);
\]

\[
\varphi_0(r, \lambda) = \frac{r^\lambda}{2^\lambda \Gamma(1 + \lambda)}.
\]

(45)

The operator \( \hat{\Delta}_0(r) \) has not discrete spectrum, so:

\[
A(\lambda) = 0
\]

so: \( \delta(\lambda) = 0 \)

(46)

Therefore, the continuum spectrum of the operator \( \hat{\Delta}_0 \) takes the form:

\[
r \cdot \delta(r - \delta) = + \frac{1}{2} \int_{-\infty}^{+\infty} \varphi_0(r, i \tau) \cdot \varphi_0^*(\rho, i \tau) \frac{\tau \cdot d\tau}{sh(\pi \tau)}.
\]

(47)

We calculate the spectrum of the operator \( \hat{D}_0(r) \); the eigenvalues equation is:

\[
\hat{D}_0(r) \varphi_0(r, \lambda) = \lambda^2 \varphi_0(r, \lambda),
\]

where \( \varphi_0(r, \lambda) = I_\lambda(r) \).

(48)

The spectral equation attached to operator \( \hat{D}_0(r) \) has the form:

\[
r \cdot \delta(r - \rho) = \sum_{n=1}^{\infty} \frac{I_n(r \cdot \rho) \mu_n(\rho)}{\mu_n(\rho)} \left[ I_n(r) \cdot \frac{\tau \cdot d\tau}{sh(\pi \tau)} - I_n(r) \cdot \frac{\tau \cdot d\tau}{sh(\pi \tau)} \right].
\]

(49)

The discrete spectrum norm has the form:

\[
\left\| I_\lambda \right\| = \int_0^\infty I_\lambda(r) \cdot \frac{dr}{r} = \frac{1}{2\lambda}.
\]

(50)

For the spectral quantities calculation \( \mu(\pm i \tau) \), will take into account that the Bessel functions \( I_\lambda(r) \) accept asymptotic solutions on the form:

\[
I_\lambda(r) = \sqrt{2 \pi \over \tau} \cdot A(\lambda) \cdot \sin \left[ r + \delta(r) - \frac{\pi}{2} \left( \lambda - \frac{1}{2} \right) \right]
\]

(51)

\[
\frac{dI_\lambda(r)}{dr} = \sqrt{2 \pi \over \tau} \cdot \cos \left[ r + \delta(r) - \frac{\pi}{2} \left( \lambda - \frac{1}{2} \right) \right].
\]

We identify solutions (50) with general asymptotic solutions which include spectral data \( A(\lambda), \delta(\lambda) \) described in (5) and (6) and which can be written in the form:

\[
\varphi(r, \lambda) = \sqrt{2 \pi \over \tau} \cdot A(\lambda) \cdot \sin \left[ r + \delta(r) - \frac{\pi}{2} \left( \lambda - \frac{1}{2} \right) \right]
\]

(52)

As a result of identification we obtained:

\[
A(\lambda) = 1
\]

\[
\delta(\lambda) = 0
\]

\[
\mu(i \tau) = \sin \left( x_0 + \frac{\pi}{2} i \tau \right)
\]

\[
\mu(-i \tau) = \sin \left( x_0 - \frac{\pi}{2} i \tau \right).
\]

(53)

\[
\lambda_n = 2n
\]

\[
\lim_{\lambda_n \to 0} \mu(-i \tau) = -1
\]

So, the symbolic representation of the spectrum of \( \hat{D}_0(r) \) takes the form:

\[
r \cdot \delta(r - \rho) = \sum_{n=1}^{\infty} 4n \cdot I_{2n}(p) \cdot I_{2n}(r) + \frac{1}{2} \int_{-\infty}^{+\infty} \left[ I_n(r) \cdot I_n(\rho) \right] \cdot \frac{\tau \cdot d\tau}{sh(\pi \tau)}.
\]

(54)

The integral equation of scattering from (38) can be written in the new conditions:

\[
A_{\lambda,\rho}(r, \rho) + f_{\lambda,\rho}(r \cdot \rho) + \int_0^z A_{\lambda,\rho}(z, r) \cdot f_{\lambda,\rho}(z \cdot \rho) \, dz = 0
\]

(55)

or otherwise written (in terms of the crossing operator \( \hat{X}_{\lambda,\rho} \)) in the form:
The spectral data (Fourier transform after $\rho$) have the form:
\[
\varphi(r, \lambda) = \int_0^{\infty} \varphi^2(r, \lambda_u) \frac{d\rho}{r} = A^2(\lambda_u) \left[ \frac{\pi}{2} - \delta'(\lambda_u) \right];
\]
\[
\delta'(\lambda_u) = \left( \frac{d\delta}{dr} \right)_{\lambda_u} \tag{65}
\]
where $A(\lambda)$ and $\delta(\lambda)$ are the spectral data from the asymptotic solutions of the wave equation.

These asymptotic solutions have the form:
\[
\sqrt{\pi r^2} \cdot \varphi(r, \lambda) \equiv A(\lambda) \cdot \sin \left[ r + \delta(\lambda) - \frac{\pi}{2} \left( \lambda - \frac{1}{2} \right) \right] \tag{66}
\]
\[
\sqrt{\pi r^2} \cdot \varphi'(r, \lambda) \equiv A(\lambda) \cdot \cos \left[ r + \delta(\lambda) - \frac{\pi}{2} \left( \lambda - \frac{1}{2} \right) \right]
\]
\[
; \varphi'(\lambda) = \left( \frac{d\varphi}{dr} \right)_{\lambda_u}. \tag{67}
\]

3. The qualitative analysis of the nonlinear equations using integral relations for solitons in photorefractive nonlinear crystals

The constitutive equations concerning the optical dynamic ($\Phi_{nj}$, $\Phi_{nj}$) have the form [31,32]:
\[
2i \frac{\partial \Phi_{nj}}{\partial \eta} + \frac{\partial^2 \Phi_{nj}}{\partial \rho^2} + \mu \cdot NL(\eta', \rho) \Phi_{nj} = 0 \tag{68}
\]
\[
2i \frac{\partial \Phi_{nj}}{\partial \eta} + \frac{\partial^2 \Phi_{nj}}{\partial \rho^2} + NL(\eta', \rho) \Phi_{nj} = 0
\]
where $\mu$ represent an asymmetry coefficient, and the nonlinear component is defined by:
\[
NL(\eta', \rho) = \frac{1}{1 + 2r^2 \gamma^2(z)} \cdot \left[ \Phi_{nj}^2 + \Phi_{nj}^2 \right] \tag{69}
\]
where $r = \frac{I_0}{I_B}$ is the ratio between the maximum intensity and background intensity, and
\[
\gamma = \frac{e^{-\frac{\mu^2}{2}}}{\cos \left[ \frac{1}{2} \left( g_0 k \cdot z - \frac{\pi}{2} + \phi \right) \right]} \tag{70}
\]
where \( \alpha \) represent an linear loss coefficient, and \( g_1 \cdot k \) nonlinear spatial frequency.

Given the Fourier transform operators defined in the form:

\[
\hat{F} = \int_{-\infty}^{\infty} e^{-2\pi i k \rho} d\rho
\]

and given the inverse Fourier transforms:

\[
\Phi_{\eta_1}(\eta', k) = \int_{-\infty}^{\infty} \Phi_{\eta_1}(\eta', \rho) e^{-2\pi i \rho k} d\rho
\]

and given the Fourier transforms:

\[
\tilde{\Phi}_{\eta_1}(\eta', k) = \hat{F} \Phi_{\eta_1}(\eta', \rho) e^{-2\pi i \rho k} d\rho
\]

we process the convolution integrals from (74) and we obtain:

\[
\begin{align*}
2i \frac{\partial \tilde{\Phi}_{\eta_1}(\eta', k)}{\partial \eta'} & = (-2\pi i k) \tilde{\Phi}_{\eta_1}(\eta', k) + \int_{-\infty}^{\infty} \hat{N}(\eta', \rho' \cdot \rho) \cdot \tilde{\Phi}_{\eta_1}(\eta', \rho') e^{-2\pi i \rho k} d\rho = 0, \\
2i \frac{\partial \Phi_{\eta_1}(\eta', k)}{\partial \eta'} & = (-2\pi i k) \Phi_{\eta_1}(\eta', k) + \int_{-\infty}^{\infty} \hat{N}(\eta', \rho' \cdot \rho) \cdot \Phi_{\eta_1}(\eta', \rho') e^{-2\pi i \rho k} d\rho = 0,
\end{align*}
\]

Thus, from the point of view of the inverse problem, \( \tilde{\Phi}_{\eta_1}(\eta', k) \) and \( \Phi_{\eta_1}(\eta', k) \) represent the spectral data of the problem. The equations (74) integrated upon \( \eta' \) represent the time and space evolution of the spectral data associated to the inverse problem. As a result of the integration upon \( \eta' \) we obtain the equations [33-36]:

\[
\begin{align*}
\tilde{\Phi}_{\eta_1}(\eta', k) & = e^{2\pi i \eta' k} \left\{ C_1(k) - \frac{\mu}{\pi} \int_{-\infty}^{\infty} \tilde{N}(\eta', \rho' \cdot k) \cdot \tilde{\Phi}_{\eta_1}(\eta', k') e^{2\pi i \eta' k'} d\rho' \right\}, \\
\Phi_{\eta_1}(\eta', k) & = e^{2\pi i \eta' k} \left\{ C_2(k) - \frac{\mu}{\pi} \int_{-\infty}^{\infty} N(\eta', \rho' \cdot k) \cdot \Phi_{\eta_1}(\eta', k') e^{2\pi i \eta' k'} d\rho' \right\}
\end{align*}
\]

where the initial (spectral) conditions \( C_1(k), \) \( C_2(k) \) have the form:

\[
\begin{align*}
\tilde{\Phi}_{\eta_1}(\eta', k) & = e^{2\pi i \eta' k} \left\{ \Phi_{\eta_1}(0, k) \right\}, \\
\Phi_{\eta_1}(\eta', k) & = e^{2\pi i \eta' k} \left\{ \Phi_{\eta_1}(0, k) \right\}
\end{align*}
\]

and \( \Phi_{\eta_1}(0, \rho) \) and \( \Phi_{\eta_1}(0, \rho) \) are the initial data of the problem.

One operates the inverse transform of the field envelopes (76) and one gets (integral solutions in the form) [37,38]:

\[
\begin{align*}
\Phi_{\eta_1}(\eta', \rho) & = e^{2\pi i \rho^2} \frac{\eta'}{i \pi} \Phi_{\eta_1}(0, \rho') + \frac{\eta'}{i \pi} \Phi_{\eta_1}(0, \rho') - \int_{\rho}^{\infty} \frac{\Phi_{\eta_1}(0, \rho')}{(\rho^2 - \rho'^2)} d\rho', \\
\Phi_{\eta_1}(\eta', \rho) & = e^{2\pi i \rho^2} \frac{\eta'}{i \pi} \Phi_{\eta_1}(0, \rho') + \frac{\eta'}{i \pi} \Phi_{\eta_1}(0, \rho') - \int_{\rho}^{\infty} \frac{\Phi_{\eta_1}(0, \rho')}{(\rho^2 - \rho'^2)} d\rho'
\end{align*}
\]

where:

\[
\begin{align*}
\Phi_{\eta_1}(\eta', \rho) & = e^{2\pi i \rho^2} \frac{\eta'}{i \pi} \Phi_{\eta_1}(0, \rho'), \\
\Phi_{\eta_1}(\eta', \rho) & = e^{2\pi i \rho^2} \frac{\eta'}{i \pi} \Phi_{\eta_1}(0, \rho')
\end{align*}
\]

The initial conditions are contained (are supplied) in the equations (78). Thus:

\[
\begin{align*}
\lim_{\eta' \to 0} \Phi_{\eta_1}(\eta', \rho') & = \lim_{\eta' \to 0} \Phi_{\eta_1}(\eta', \rho') = \Phi_{\eta_1}(0, \rho'), \\
\lim_{\eta' \to 0} \Phi_{\eta_1}(\eta', \rho') & = \lim_{\eta' \to 0} \Phi_{\eta_1}(\eta', \rho') = \Phi_{\eta_1}(0, \rho').
\end{align*}
\]

We define the function:

\[
N(\eta', \rho') = \frac{1}{\pi} \frac{1}{2 \pi \eta'} + \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{N}(\eta', \rho') e^{-2\pi i \eta' \rho'} d\rho'
\]

Example of calculation:

We assume that the pump is accomplished by a Gaussian beam, so that we will have:

\[
\begin{align*}
\Phi_{\eta_1}(0, \rho') & = \frac{1}{\sqrt{\pi} \cdot \sigma_x} \cdot e^{-\rho'^2/4 \sigma_x^2}, \\
\Phi_{\eta_1}(0, \rho') & = \frac{1}{\sqrt{\pi} \cdot \sigma_y} \cdot e^{-\rho'^2/4 \sigma_y^2}
\end{align*}
\]

where: \( \sigma_{x,y} = \text{Re}(\sigma_{x,y}) + i \cdot \text{Im}(\sigma_{x,y}) \), and \( \sigma_{x,y} \) belong to the complex numbers class \( (\sigma_{x,y} \in \mathbb{C}) \).
We will have:
\[
\Phi_{x_0} = \frac{1}{\sqrt{\pi} \sigma_{x_0}} \int_{-\infty}^{\infty} e^{\frac{(\rho-\rho')^2}{2\sigma_{x_0}^2}} d\rho'.
\]  
(82)

In these conditions, after integrating (81) we obtain:
\[
\Phi_{x_0}(\eta, \rho) = \frac{e^{-\frac{\rho^2}{\sigma_{x_0}^2+2i\eta'}}}{\sqrt{\pi}\left(\sigma_{x_0}^2 + 2i\eta'\right)}. \tag{83}
\]

We use the case of the circular symmetry ($\sigma_x = \sigma_y = \sigma_L = \sigma_{LR} + i\sigma_{LZ}$). Thus, we will have:
\[
\Phi_{x_0}(0, \rho') = \Phi_{y_0}(0, \rho') = \frac{1}{\sqrt{\pi} \cdot \sigma_L} \cdot \frac{e^{-\frac{\rho'^2}{\sigma_L^2}}}{\sqrt{\pi}\left(\sigma_L^2 + 2i\eta'\right)}. \tag{84}
\]

\[
\Phi_{x_0}(\eta', \rho') = \Phi_{y_0}(\eta', \rho') = \frac{e^{-\frac{\rho'^2}{\sigma_L^2+2i\eta'\gamma}}}{\sqrt{\pi}\left(\sigma_L^2 + 2i\eta'\gamma\right)}. \tag{85}
\]

In the integral equations, the nonlinear function (nonlinear nucleus) and only that is written in the form:
\[
NL(\eta^*, \rho') = \frac{1}{1 + \frac{4r}{\pi} \cdot \eta^2 \cdot \Phi_{x_0}^2(\eta^*, \rho')} \tag{86}
\]

From the symmetry of the optical field envelopes it results:
\[
\Phi_{y_1}(\eta', \rho) = \Phi_{x_1}(\eta', \rho) \bigg|_{\mu=1}. \tag{87}
\]

So, from now on, we will effect the calculus on $\Phi_{x_1}$, and $\Phi_{x_1} = \Phi_{x_0} \bigg|_{\mu=1}$, Thus, the expression of $\Phi_{x_1}$ yields:
\[
\Phi_{x_1}(\eta', \rho) = \Phi_{x_0}(\eta', \rho) - \frac{\mu}{2i} \cdot D_x, \tag{88}
\]

where $\Phi_{x_0}$ and $D_x$ are asymptotic forms for $\Phi_{x_1}$ and $D_x$. Thus, for $D_{x_0}$ it results the expression:
\[
D_{x_0} = \int_0^\eta d\eta' \frac{\Phi_{x_0}(\eta', \rho')}{\sqrt{\pi\left[\frac{4r}{\pi\gamma^2} \cdot \Phi_{x_0}^2(\eta', \rho') + 2i\eta'(\eta' - \eta)\right]}}. \tag{89}
\]

Using the approximation:
\[
\int_0^\eta d\eta' \rightarrow \eta', \tag{90}
\]

and crossing to the limit $\eta^* \rightarrow \eta'$ result the algebraic form of $D_{x_0}$:
\[
D_{x_0} = \frac{1}{4} \cdot \frac{\pi}{\gamma} \cdot \eta' \cdot \left[ 1 + \frac{4r}{\pi\gamma^2} \right] \left[ \frac{\gamma}{\sigma_L^2 + 2i\eta'} \right] \tag{91}
\]

Thus, it results that the envelope solutions are corresponding each to another:
\[
\Phi_{x_1}(\eta', \rho) = \Phi_{x_0}(\eta', \rho) - \frac{\mu}{2i} \cdot D_x, \tag{92}
\]

We enumerate a few properties of the envelope amplitudes:
\[
\lim_{\gamma \rightarrow 0} \Phi_{x_1}(\eta', \rho) = \Phi_{x_0}(\eta', \rho) \cdot \left[ 1 - \frac{\mu}{2i} \eta' \right] \tag{93}
\]

The pure solitonic solution condition (as additive term and $\sigma_L$ - complex) results under the form:
\[
\gamma(\eta') = \frac{\pi \sqrt{\sigma_L^2 + 2i\eta'}}{4r}, \tag{94}
\]

and the $D_{x_0}$ expression becomes [36,37]:
The inverse problem in scattering theory of optical fields

The soliton width, $\Delta \rho$, will be defined as:

$$\Delta \rho = \frac{2}{\pi} \gamma(\eta') \cdot \sqrt{r}.$$  (98)

But, from the link relation (96) and the relation (98) it results from the initial condition for $\eta'(\gamma)$:

$$\gamma(\eta') = \frac{\pi}{2r} \cdot \sigma_L$$  (with $\sigma_L$ real)  (99)

for the function:

$$D_s = \frac{\eta'}{2\sqrt{\pi} \Delta \rho} \cdot \cosh \left( \frac{\rho}{\Delta \rho} \right).$$  (100)

If we calculate the optical soliton width, $\Delta \rho$, at $\frac{1}{2}$ of maximum amplitude, it results [39,40]:

$$\Delta \rho = A \cdot \left( \sqrt{B - \sqrt{C}} + \frac{1}{4\sqrt{B}} \cdot \ln r + \frac{1}{4\sqrt{C}} \cdot \ln \frac{1}{r} \right).$$  (101)

where:

$$A = \sqrt{\sigma_L^2 + 2n\eta}; \quad B = \ln \left[ \frac{2 + \sqrt{3}}{\pi \cdot A} \right];$$

$$C = \ln \left[ \frac{2 - \sqrt{3}}{\pi \cdot A} \right].$$  (102)

If we linearize the expression of $\Delta \rho$ from (100), comes out the dependence:

$$\Delta \rho = \frac{A}{2\sqrt{B}} \cdot r + \frac{A}{2\sqrt{C}} \cdot \frac{1}{r} + A \cdot \left( \sqrt{B - \sqrt{C}} - \frac{3}{8} \frac{\sqrt{B} + \sqrt{C}}{\sqrt{BC}} \right).$$  (103)

wherefrom results an important quantity for the minimal soliton width. Thus:

$$\Delta \rho(r = 1) = \frac{A}{8} \left[ \frac{B + C + 2\sqrt{BC} \cdot [1 + 4(B - C)]}{B \cdot \sqrt{C} + \sqrt{B \cdot C}} \right].$$  (104)

and

$$\Delta \rho_{\min} = A \cdot \left[ \frac{1}{\sqrt{BC}} + \left( \sqrt{B - \sqrt{C}} \right) - \frac{3}{8} \frac{\sqrt{B} + \sqrt{C}}{\sqrt{BC}} \right].$$  (105)

The dependence $\Delta \rho(r)$ is presented in the Fig. 1.

Fig. 1. The dependence of the optical soliton width depending on the parameter $r = \frac{1}{\sqrt{1-B}}$.

4. Conclusions

In this paper was presented the spectral theory of the nonlinear operators with applications to the inverse scattering theory (the Gelfand-Levitan-Marcenko theory) concerning the screening spatial optical solitons theory. Analytically, there was evaluated the strength of the screening of spatial solitons in photorefractive crystals, using specific elements from the inverse problem of the scattering theory (at constant energy) after P. Sabatier.

References


