

A complete solution of the mathematical problem for the behaviour of the flexoelectric domains in a d.c. voltage for the case of anisotropic elasticity*

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The solution of the Euler-Lagrange equations for the director components $n_y=f_1(z)\sin qy$ and $n_z=f_2(z)\cos qy$, where q is the wave number of the flexoelectric domains of Vistin'-Pikin-Bobylev, has been for the first time exactly found with the aid of matrix calculations for the case of a planar nematic layer with anisotropic elasticity and a negative dielectric anisotropy under the action of an inhomogeneous d.c. flexoelectrically deforming electric field. A comparison is made with another, approximate, solution for anisotropic elasticity and a homogeneous electric field. A discussion of the eventual applications of this solution is also presented.

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1. Introduction

DC voltage-induced static domains oriented along the initial alignment of the nematic director \mathbf{n} with an electrically controlled period have been observed for the first time by Vistin' [1]. This author has observed that the period of the domains decreases with increasing voltage. Bobylev and Pikin [2] first proved the flexoelectric nature of these domains, by developing a theory for the case of isotropic elasticity. They obtained simple relations for the threshold voltage U_c and the wave number q_c of the flexoelectric domains (details can be found in the review by Pikin [3]). Their simple theory was later extended by Bobylev, Chigrinov and Pikin [4] and by Pikin [5,6], for the case of anisotropic elasticity and an equal exponential dependence along z of both components of the director. New theories [7-9] have considered the influence of the flexoelectric effect on the thermal fluctuations of the nematic director [10] and shed new light on the appearance and development of the flexoelectric domains of Vistin'-Pikin-Bobylev. The authors of these theories have used the matrix analysis [11,12].

In this paper, we present the complete solution of the problem based on the theoretical results of Romanov and Sklyarenko for our concrete case (some of the detailed matrix calculations are given elsewhere [13]). Let us first mention that the flexoelectric term due to eventual

inhomogeneity of the electric field $(e_{1z}+e_{3x})(dE/dz)$ was included in the final solution of the problem under consideration.

2. Theory and results

The minimization of the "electric enthalpy" with respect to the director components n_y and n_z and their derivatives with respect to the coordinates y and z yields two equations of Euler-Lagrange. Performing the calculations, taking into account that \mathbf{n} is a unit vector and accepting that

$$n_y = f_1(z) \sin qy, n_z = f_2(z) \cos qy$$

(see References [2], [4], [8] and [14]), finally we have obtained the following two equations for the unknown arbitrary functions $f_1(z)$ and $f_2(z)$ which we represent in a matrix form in the following way:

$$\left(\hat{\mathbf{A}}_0 + \hat{\mathbf{C}} \frac{d}{dz} + \hat{\mathbf{D}} \frac{d^2}{dz^2} \right) \hat{\mathbf{f}} = 0 \quad (1)$$

The matrices $\hat{\mathbf{A}}_0$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$ and $\hat{\mathbf{f}}$ can be represented in the following forms:

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$$\hat{\mathbf{A}}_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\hat{\mathbf{C}} = \begin{vmatrix} 0 & (K_{11} - K_{22})q \\ -(K_{11} - K_{22})q & 0 \end{vmatrix}$$

$$\hat{\mathbf{D}} = \begin{vmatrix} K_{11} & 0 \\ 0 & K_{22} \end{vmatrix}, \hat{\mathbf{f}} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$a_{11} = -K_{22}q^2 - (e_{1z} + e_{3x}) \frac{dE}{dz} - \frac{|\Delta\varepsilon|}{4\pi} E^2$$

$$a_{12} = a_{21} = (e_{1z} - e_{3x})Eq, \quad a_{22} = -K_{11}q^2$$

The sum of the operators in Eq. (1) cannot be presented in a diagonal form by a transformation of similarity, since it is not self-conjugated. Consequently, we must transform Eq. (1) so that the left-hand side contains an operator admitting diagonalization. For this purpose, following Romanov and Skljarenko we transform Eq. (1) as follows: First, we multiply Eq. (1) from the left by the inverse matrix $\hat{\mathbf{D}}^{-1}$:

$$\left(\hat{\mathbf{D}}^{-1} \hat{\mathbf{A}}_0 + \hat{\mathbf{D}}^{-1} \hat{\mathbf{C}} \frac{d}{dz} + (\hat{\mathbf{D}}^{-1} \hat{\mathbf{D}}) \frac{d^2}{dz^2} \right) \hat{\mathbf{f}} = 0 \quad (2)$$

The inverse matrix $\hat{\mathbf{D}}^{-1}$ has the following form:

$$\hat{\mathbf{D}}^{-1} = \begin{vmatrix} 1 & 0 \\ K_{11} & 1 \\ 0 & K_{22} \end{vmatrix}$$

As a second step, we multiply the term in Eq. (2) which is in parentheses by the following term:

$$e^{-\frac{\hat{\mathbf{D}}\hat{\mathbf{C}}}{2}z} e^{\frac{\hat{\mathbf{D}}\hat{\mathbf{C}}}{2}z} = 1$$

Introducing the new variable: $\hat{\mathbf{f}}_1 = e^{\frac{\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}}{2}z} \hat{\mathbf{f}}$ and

performing the differentiation with respect to z in $e^{-\frac{\hat{\mathbf{D}}\hat{\mathbf{C}}}{2}z}$, we have obtained the following operator equation:

$$\left\{ \hat{\mathbf{H}} e^{-\frac{\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}}{2}z} + e^{-\frac{\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}}{2}z} \hat{\mathbf{I}} \frac{d^2}{dz^2} \right\} \hat{\mathbf{f}}_1 = 0 \quad (3)$$

$$\hat{\mathbf{H}} = \hat{\mathbf{D}}^{-1} \hat{\mathbf{A}}_0 - \frac{1}{4} (\hat{\mathbf{D}}^{-1} \hat{\mathbf{C}})^2$$

Finally, multiplying Eq. (3) from the left by $e^{\frac{\hat{\mathbf{D}}\hat{\mathbf{C}}}{2}z}$, we have obtained another operator equation:

$$\left(\hat{\mathbf{H}}_1 + \hat{\mathbf{I}} \frac{d^2}{dz^2} \right) \hat{\mathbf{f}}_1 = 0 \quad (4)$$

$$\hat{\mathbf{H}}_1 = e^{\frac{\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}}{2}z} \hat{\mathbf{H}} e^{-\frac{\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}}{2}z}$$

The matrices of rotation are connected with the anisotropy of the elasticity [9], [14]:

$$e^{\pm \frac{\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}}{2}z} = \begin{vmatrix} \cos \alpha & \pm \sqrt{\frac{K_{22}}{K_{11}}} \sin \alpha \\ \mp \sqrt{\frac{K_{11}}{K_{22}}} \sin \alpha & \cos \alpha \end{vmatrix} \quad (5)$$

$$\alpha = \frac{1}{2} \frac{K_{11} - K_{22}}{\sqrt{K_{11}K_{22}}} qz$$

(detailed calculations are given elsewhere [13]).

The next step in the calculations is the introduction of a new matrix $\hat{\mathbf{B}}_q^p$, which can be made diagonal:

$$\hat{\mathbf{B}}_q^p = \hat{\mathbf{H}}_1 + \hat{\mathbf{I}} \frac{d^2}{dz^2} \quad (6)$$

$$\hat{\mathbf{B}}_q^p = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$b_{11} = -a \cos^2 \alpha - b \sin^2 \alpha + f \sin(2\alpha) + \frac{d^2}{dz^2}$$

$$b_{12} = \sqrt{\frac{K_{22}}{K_{11}}} \left(\frac{a-b}{2} \sin(2\alpha) + f \cos(2\alpha) \right),$$

$$b_{21} = \sqrt{\frac{K_{11}}{K_{22}}} \left(\frac{a-b}{2} \sin(2\alpha) + f \cos(2\alpha) \right),$$

$$b_{22} = -a \sin^2 \alpha - b \cos^2 \alpha - f \sin(2\alpha) + \frac{d^2}{dz^2}$$

$$a = \left(\frac{1}{2} + \frac{3}{4} \frac{K_{22}}{K_{11}} - \frac{1}{4} \frac{K_{11}}{K_{22}} \right) q^2$$

$$+ \frac{(e_{1z} + e_{3x})}{K_{11}} \frac{dE}{dz} + \frac{|\Delta\varepsilon|}{4\pi K_{11}} E^2$$

$$b = \left(\frac{1}{2} + \frac{3}{4} \frac{K_{11}}{K_{22}} - \frac{1}{4} \frac{K_{22}}{K_{11}} \right) q^2$$

$$f = \frac{(e_{1z} - e_{3x})}{\sqrt{K_{11}K_{22}}} Eq$$

where the index q is obvious and the index p means "planar". Multiplying the matrix $\hat{\mathbf{B}}_q^p$ from the left by the inverse matrix $\hat{\mathbf{V}}^{-1}$ of the eigenvectors of $\hat{\mathbf{B}}_q^p$ and the matrix $\hat{\mathbf{f}}_1$ by $\hat{\mathbf{V}} \hat{\mathbf{V}}^{-1} = 1$, we obtain:

$$\hat{\mathbf{V}}^{-1} \hat{\mathbf{B}}_q^p \hat{\mathbf{V}} \hat{\mathbf{V}}^{-1} \hat{\mathbf{f}}_1 = 0 \quad (7)$$

where

$$\hat{\mathbf{V}} = \begin{vmatrix} (w-p)\sqrt{\frac{K_{22}}{K_{11}}} & (w+p)\sqrt{\frac{K_{22}}{K_{11}}} \\ -u & -u \end{vmatrix}$$

$$w = -a \sin^2 \alpha - b \cos^2 \alpha - f \sin(2\alpha) + \frac{a+b}{2}$$

$$p = \sqrt{\left(\frac{a-b}{2}\right)^2 + f^2}$$

$$u = -\left(\frac{b-a}{2} \sin(2\alpha) - f \cos(2\alpha)\right)$$

(detailed calculations are given elsewhere [13]).

Introducing $\hat{\mathbf{V}}^{-1} \hat{\mathbf{f}}_1 = \hat{\mathbf{f}}_2$ and $\hat{\mathbf{W}} = \hat{\mathbf{V}}^{-1} \hat{\mathbf{B}}_q^p \hat{\mathbf{V}}$, we obtain:

$$\hat{\mathbf{W}} \begin{vmatrix} f_{21}(z) \\ f_{22}(z) \end{vmatrix} = 0 \quad (8)$$

The diagonal matrix $\hat{\mathbf{W}}$ has the following form:

$$\hat{\mathbf{W}} = \begin{vmatrix} -Q^2 + \frac{d^2}{dz^2} & 0 \\ 0 & -P^2 + \frac{d^2}{dz^2} \end{vmatrix} \quad (9)$$

Going back to the matrix $\hat{\mathbf{f}}$ via the following transformations:

$$\hat{\mathbf{V}} \hat{\mathbf{V}}^{-1} \hat{\mathbf{f}}_1 = \hat{\mathbf{V}} \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_1 = \hat{\mathbf{V}} \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_1 e^{\frac{\hat{\mathbf{D}}^{-1} \hat{\mathbf{C}}}{2} z} = \hat{\mathbf{f}} \quad (10)$$

we have obtained finally the solution for the two functions $f_1(z)$ and $f_2(z)$:

$$f_1(z) = \sqrt{\frac{K_{22}}{K_{11}}} \{[(w-p)f_{21}(z) + (w+p)f_{22}(z)] \cos \alpha + u \sin \alpha [f_{21}(z) + f_{22}(z)]\} \quad (11)$$

$$f_2(z) = [(w-p)f_{21}(z) + (w+p)f_{22}(z)] \sin \alpha - u \cos \alpha [f_{21}(z) + f_{22}(z)]$$

where $f_{21}(z)$ and $f_{22}(z)$ are the solutions of the following two equations:

$$\left(-Q^2(z) + \frac{d^2}{dz^2}\right) f_{21}(z) = 0$$

$$\left(-P^2(z) + \frac{d^2}{dz^2}\right) f_{22}(z) = 0 \quad (12)$$

$$Q = \sqrt{\frac{a+b}{2} - \sqrt{\left(\frac{a-b}{2}\right)^2 + f^2}}$$

$$P = \sqrt{\frac{a+b}{2} + \sqrt{\left(\frac{a-b}{2}\right)^2 + f^2}}$$

Let us represent Q in the following more convenient form:

$$Q = \sqrt{\frac{a+b}{2} - \sqrt{\left(\frac{a-b}{2} + f\right)^2 - f(a-b)}} \quad (13)$$

Since

$$\left(\frac{a-b}{2} + f\right)^2 \gg f(a-b)$$

$$(a-b) \approx d^{-2} \gg 1, f \approx d^{-2} \gg 1 \quad (14)$$

(where d is the thickness of the liquid crystal cell)

$$Q^2 \cong b - f \quad (15)$$

Similarly,

$$P^2 \cong a + f \quad (16)$$

We can solve the equations (12) exactly or approximately, accepting a certain form of the electric field: linear, hyperbolic, exponential, etc. and the approximate values of P and Q .

2. Conclusions

The solution of the problem concerning the flexoelectric domains with anisotropic elasticity is not only a mathematical one. Firstly, it clarifies the influence of the different physical parameters on the creation and

development of the flexoelectric domains. For instance, the threshold characteristics such as the threshold voltage U_c and the threshold wave number q_c in the case of strong anchoring have been obtained [15,16]. These results unambiguously show that there is a limit in the ratio K_{22}/K_{11} permitting the development of the flexoelectric domains. In contrast, the anisotropic approximate theory [4] does not show such a limit.

Secondly, the flexoelectric term, depending on the inhomogeneity of the electric field $(e_{1z}+e_{3x})(dE/dz)$, was included in the final solution of the problem under consideration.

In general, the complete solution of the problem can find application in more complex cases, such as the development of the flexoelectric domains in simultaneously applied d.c. and a.c. voltages., weak anchoring, strong-weak anchoring, etc.

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