# Bright and dark optical solitons in birefringent fibers with Hamiltonian perturbations and Kerr law nonlinearity

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This paper obtains the 1-soliton solution of birefringent fibers, with Kerr law nonlinearity, in presence of perturbation terms. The perturbation terms that are studied are the inter-modal dispersion, third order dispersion, self-steepening term and nonlinear dispersion. Both bright and dark soliton solutions are considered. There are several constraint conditions that fall out during the course of derivation of the soliton solutions. The numerical simulations are also provided.

(Received December 17, 2011; accepted July 19, 2012)

*Keywords:* 1-soliton solution of birefringent fibers, Kerr law nonlinearity, Bright and dark soliton solutions

## 1. Introduction

Optical solitons is a very important area of research in the area of Nonlinear Optics. Several papers are published in this area for the past few decades [1-15]. There has been an overwhelming amount of sequential new results that are constantly being reported in several journals. While the focus in most of the existing papers is on polarization preserving fibers, it is also equally important to study birefringent fibers in details. The integrability aspects of solitons in birefringent fibers, in presence of strong Hamiltonian perturbations, is the focus of study in this paper. In order to be complete, both bright and dark soliton solutions are covered in this paper.

Optical soliton solutions in birefringent fibers requires special analytic as well as numerical treatment to carry out its investigation. As the polarization state of the optical wave changes with propagation distance, the interaction between two linear polarization states is obvious. The optical wave energy interchanged in different polarization state depends on the birefringence value [1, 2, 5, 6]. For strong birefringent materials such as liquid crystals, this interaction is observed within a very short distance. On the other hand, weak birefringence that is observed in regular optical fibers, this interaction between polarization states is gradual [5, 6]. Solving nonlinear Schrödinger's equation (NLSE) usually gives a soliton solution for the wave propagation in an optical fiber [2, 5]. However, for birefringent fibers, soliton solution exists if the system satisfies some constraint conditions.

The general structure of the NLSE in birefringent fibers with strong Hamiltonian type perturbation taken into consideration is given by

$$iq_t + a_1 q_{xx} + \left\{ b_1 F\left(|q|^2\right) + c_1 F\left(|r|^2\right) \right\} q + i\alpha_1 q_x + i\lambda_1 \left(|q|^2 q\right)_x + i\nu_1 \left(|q|^2\right)_x q + i\gamma_1 q_{xxx} = 0 \quad (1)$$

$$ir_t + a_2 r_{xx} + \left\{ b_2 F\left(|r|^2\right) + c_2 F\left(|q|^2\right) \right\} r + i\alpha_2 r_x + i\lambda_2 \left(|q|^2 q\right)_x + i\nu_2 \left(|r|^2\right)_x r + i\gamma_2 q_{xxx} = 0 \quad (2)$$

Here, in (1) and (2), the function $F$ represents, in general, the non-Kerr law nonlinearity. The independent variables $x$ and $t$ represents the spatial and temporal variables respectively. The dependent variables $q(x,t)$ and $r(x,t)$ represents the optical wave profile. The constant coefficients $a_l$, $b_l$, $c_l$, $\lambda_l$ and $\gamma_l$ for $l = 1, 2$ respectively represent the coefficients of group velocity dispersion (GVD), self-phase modulation, cross-phase modulation, self-steepening terms and third order dispersion (TOD) terms for the two polarized pulses. Additionally, $\alpha_l$ and $\nu_l$ represents the coefficients of inter-modal dispersion and nonlinear dispersion. It needs to be noted that for $\alpha_l = \nu_l = \lambda_l = \gamma_l = 0$, the pair (1)-(2) reduces to the Manakov model [3]. Here, in (1) and (2), the coefficients with $\alpha_l$, $\gamma_l$, $\lambda_l$ and $\nu_l$ all represent strong Hamiltonian type perturbations.

## 2. Mathematical analysis

In this section, equations (1) and (2) will be considered in the Kerr law medium where $F(s) = s$, which means that the refractive index of light is directly proportional to its intensity. Thus, in a Kerr law medium, the circularly polarized components of the electric fields $q$ and $r$ is given by the following set of coupled generalized NLSE in dimensionless form [5, 6]:

$$iq_t + a_1 q_{xx} + \left(b_1 |q|^2 + c_1 |r|^2\right) q + i\alpha_1 q_x + i\lambda_1 \left(|q|^2 q\right)_x + i\nu_1 \left(|q|^2\right)_x q + i\gamma_1 q_{xxx} = 0 \quad (3)$$

$$ir_t + a_2 r_{xx} + \left(b_2 |r|^2 + c_2 |q|^2\right) r + i\alpha_2 r_x + i\lambda_2 \left(|q|^2 q\right)_x + i\nu_2 \left(|r|^2\right)_x r + i\gamma_2 q_{xxx} = 0 \quad (4)$$

The analytical solution of these two equations is shown in details in the rest of the section and in the following two subsections. The numerical simulations will also confirm the propagation of soliton through birefringent fiber that was obtained using similar numerical simulation as was reported earlier [1, 2].

In order to solve (3) and (4) for an exact solution the following hypothesis is considered.

$$q(x,t) = P_1(x,t)\, e^{i\phi_1} \quad (5)$$

$$r(x,t) = P_2(x,t)\, e^{i\phi_2} \quad (6)$$

where $P_l(x,t)$ represents the wave form which could be either a dark or a bright soliton and $\phi_l$ represents the phase components of the two pulses. Thus,

$$\phi_l = -\kappa_l x + \omega_l t + \theta_l \quad (7)$$

for $l = 1, 2$. Here in (7), $\kappa_l$ represents the frequency of the two solitons, $\omega_l$ are the wave numbers and $\theta_l$ are the phase constants.

Substituting (5) and (6) reduces (3) and (4) respectively to

$$\begin{aligned} & i\frac{\partial P_l}{\partial t} - \omega_l P_l + a_l \left( \frac{\partial^2 P_l}{\partial x^2} - 2i\kappa_l \frac{\partial P_l}{\partial x} - \kappa_l^2 P_l \right) \\ & + b_l P_l^3 + c_l P_l P_{\bar{l}}^2 + i\lambda_l \left( 3P_l^2 \frac{\partial P_l}{\partial x} - i\kappa_l P_l^3 \right) \\ & + i\gamma_l \left( \frac{\partial^3 P_l}{\partial x^3} - 3i\kappa_l \frac{\partial^2 P_l}{\partial x^2} - 3\kappa_l^2 \frac{\partial P_l}{\partial x} + i\kappa_l^3 P_l \right) \\ & + i\alpha_l \left( \frac{\partial P_l}{\partial x} - i\kappa_l P_l \right) + 2i\nu_l P_l^2 \frac{\partial P_l}{\partial x} = 0 \end{aligned} \quad (8)$$

for $l = 1, 2$ and $\bar{l} = 3 - l$. Now, from (8), the real and imaginary part equations respectively are

$$-\left(\omega_l + \alpha_l \kappa_l + a_l \kappa_l^2 + \gamma_l \kappa_l^3\right) P_l + \left(b_l + \lambda_l \kappa_l\right) P_l^3 + c_l P_l P_{\bar{l}}^2 + \left(a_l + 3\gamma_l \kappa_l\right) \frac{\partial^2 P_l}{\partial x^2} = 0 \quad (9)$$

and

$$\frac{\partial P_l}{\partial t} + \left(\alpha_l - 2a_l\kappa_l - 3\gamma_l\kappa_l^2\right)\frac{\partial P_l}{\partial x} + \left(3\lambda_l + 2\nu_l\right) P_l^2 \frac{\partial P_l}{\partial x} + \gamma_l \frac{\partial^3 P_l}{\partial x^3} = 0 \quad (10)$$

The study will now be split into bright and dark solitons separately that will be now seen in the following two subsections.

### 2.1. Bright solitons

For bright solitons,

$$P_l(x,t) = A_l\, sech^{\,p_l} \tau\, e^{i\phi_l} \quad (11)$$

for $l = 1, 2$, where $A_l$ represents the amplitude of the soliton. Also,

$$\tau = B(x - vt) \quad (12)$$

where $B$ is the width of the two pulses and $v$ is the velocity with which the two polarized pulses travel.

Now, the real part equation that is given by (9) reduces to

$$\left\{\begin{array}{l}\left(\omega_l + a_l\kappa_l^2 + \gamma_l\kappa_l^3 - \alpha_l\kappa_l\right)A_l + \\ \left(a_l + 3\gamma_l\kappa_l\right)p_l^2 A_l B^2\end{array}\right\} sech^{p_l}\tau$$
$$-p_l(p_l+1)\left(a_l + 3\gamma_l\kappa_l\right)A_l B^2 sech^{p_l+2}\tau$$
$$-c_l A_l A_{\bar{l}}^2 sech^{p_l+2p_{\bar{l}}}\tau + \left(b_l + \lambda_l\kappa_l\right)A_l^3 sech^{3p_l}\tau = 0 \quad (13)$$

while the imaginary part equation given by (10) reduces to

$$\left\{p_l v A_l B - \left(\alpha_l - 2a_l\kappa_l + 3\gamma_l\kappa_l^2\right)p_l A_l B - \gamma_l p_l^3 A_l B^3\right\}$$
$$sech^{p_l}\tau\tanh\tau$$
$$-\left(3\lambda_l + 2\nu_l\right)p_l A_l^3 B sech^{3p_l}\tau\tanh\tau$$
$$+\gamma_l p_l(p_l+1)(p_l+2)A_l B^3 sech^{p_l+2}\tau\tanh\tau = 0 \quad (14)$$

From (14), equating the exponents $3p_1$ and $p_1+2$ yields

$$3p_l = p_l + 2 \quad (15)$$

which leads to

$$p_l = 1 \quad (16)$$

for $l = 1,2$. Also, from (14) setting the coefficients of the linearly independent functions $sech^{p_l+j}\tau\tanh\tau$ for $j = 0,2$ to zero gives

$$v = \alpha_l - 2a_l\kappa_l - 3\gamma_l\kappa_l^2 + \gamma_l B^2 \quad (17)$$

and

$$\left(3\lambda_l + 2\nu_l\right)A_l^2 = 6\gamma_l B^2 \quad (18)$$

From (10), equating the exponent pairs $p_l + 2p_{\bar{l}}$, $3p_l$ and $p_l+2$, $3p_l$ also leads to the same value of $p_l$ as in (16). Similarly, from (13), setting the coefficients of the linearly independent functions to zero yields

$$\omega_l = \left(a_l + 3\gamma_l\kappa_l\right)B^2 - \left(a_l\kappa_l^2 + \alpha_l\kappa_l + \gamma_l\kappa_l^3\right) \quad (19)$$

and

$$\left(b_l + \lambda_l\kappa_l\right)A_l^2 + c_l A_{\bar{l}}^2 = 2\left(a_l + 3\gamma_l\kappa_l\right)B^2 \quad (20)$$

Now, equating the two values of the velocity ($v$) of the solitons from (17) for $l = 1,2$ gives the width of the soliton as

$$B = \left[\frac{\left(\alpha_2 - \alpha_1\right) - 2\left(a_1\kappa_1 - a_2\kappa_2\right) + 3\left(\gamma_1\kappa_1^2 - \gamma_2\kappa_2^2\right)}{\gamma_1 - \gamma_2}\right]^{\frac{1}{2}} \quad (21)$$

which stays valid as long as

$$(\gamma_1 - \gamma_2)\left\{\begin{array}{l}\left(\alpha_2 - \alpha_1\right) - 2\left(a_1\kappa_1 - a_2\kappa_2\right) \\ +3\left(\gamma_1\kappa_1^2 - \gamma_2\kappa_2^2\right)\end{array}\right\} > 0 \quad (22)$$

From (18), the amplitudes of the two polarized solitons are given by

$$A_l = \left[\frac{6\gamma_l\left\{\left(\alpha_{\bar{l}} - \alpha_l\right) - 2\left(a_l\kappa_l - a_{\bar{l}}\kappa_{\bar{l}}\right) + 3\left(\gamma_l\kappa_l^2 - \gamma_{\bar{l}}\kappa_{\bar{l}}^2\right)\right\}}{\left(3\lambda_l + 2\nu_l\right)(\gamma_l - \gamma_{\bar{l}})}\right]^{\frac{1}{2}} \quad (23)$$

where $l = 1,2$ and $\bar{l} = 3 - l$. This forces, after taking (22) into consideration, another constraint relation pair given by

$$\gamma_l\left(3\lambda_l + 2\nu_l\right) > 0 \quad (24)$$

Moreover, (20) produce an additional pair of constraints, namely

$$\frac{3\gamma_l\left(b_l + \lambda_l\kappa_l\right)}{3\lambda_l + 2\nu_l} + \frac{3c_l\gamma_{\bar{l}}}{3\lambda_{\bar{l}} + 2\gamma_{\bar{l}}} = a_l + 3\gamma_l\kappa_l \quad (25)$$

which is obtained by substituting the polarized amplitudes $A_1$ and $A_2$ and the width $B$ into them.

In conclusion, the 1-soliton solution in birefringent fibers is given by (11) where the amplitudes of the polarized solitons are given by (23) while the width is seen in (21). The velocity of the solitons is seen in (17). The wave numbers of the solitons are given by (19). In order for these solitons to exist there are several constraint conditions that must be valid. These are (22), (24) and (25) for $l = 1,2$.

The following numerical simulation of a right soliton shows that the solitons propagate as long as the perturbation terms stay within a certain bound. For example if the balance of GVD and TOD breaks down, then the media will no longer support the soliton. It rather will start shedding dispersive energy in the form of soliton radiation.

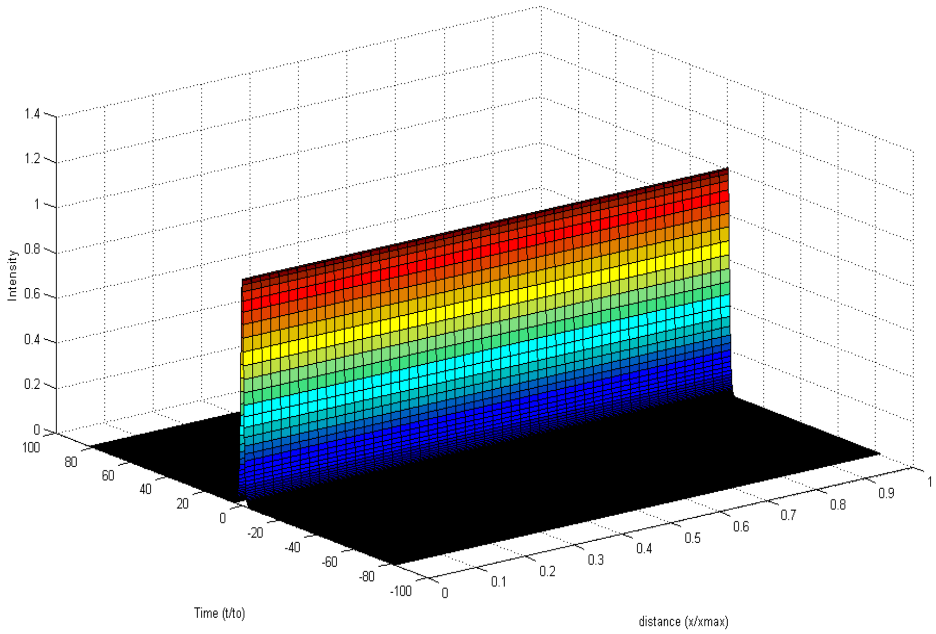


*Fig. 1. The typical shape of bright soliton.*

### 2.2. Dark Solitons

For dark solitons

$$P_l(x,t) = A_l \tanh^{p_l}\tau e^{i\phi_l} \tag{26}$$

where the definition of $\tau$ stays the same as in (12). For dark solitons, the parameters $A_l$ and $B$ represents free parameters. In this case, the real and imaginary part equations (9) and (10) respectively reduce to

$$\begin{aligned}&-\left(\omega_l-\alpha_l\kappa_l+a_l\kappa_l^2+\gamma_l\kappa_l^3\right)\tanh^{p_l}\tau\\ &+p_lB^2\left(a_l+3\gamma_l\kappa_l\right)\begin{Bmatrix}(p_l-1)\tanh^{p_l-2}\tau\\ -2p_l\tanh^{p_l}\tau+(p_l+1)\tanh^{p_l+2}\tau\end{Bmatrix}\\ &+c_lA_{\bar{l}}^2\tanh^{p_l+2p_{\bar{l}}}\tau+\left(b_l+\lambda_l\kappa_l\right)A_l^2\tanh^{3p_l}\tau=0\end{aligned} \tag{27}$$

and

$$\begin{aligned}&-p_lv\left(\tanh^{p_l-1}\tau-\tanh^{p_l+1}\tau\right)\\ &+p_l\left(\alpha_l-2a_l\kappa_l-3\gamma_l\kappa_l^2\right)\left(\tanh^{p_l-1}\tau-\tanh^{p_l+1}\tau\right)\\ &+\left(3\lambda_l+2\nu_l\right)p_lA_l^2\left(\tanh^{3p_l-1}\tau-\tanh^{3p_l+1}\tau\right)\\ &+\gamma_lp_lB^2\begin{Bmatrix}(p_l-1)(p_l-2)\tanh^{p_l-3}\tau-\\ \left(3p_l^2-3p_l+2\right)\tanh^{p_l-1}\tau\\ +\left(3p_l^2+3p_l+2\right)\tanh^{p_l+1}\tau-\\ (p_l+1)(p_l+2)\tanh^{p_l+3}\tau\end{Bmatrix}=0\end{aligned} \tag{28}$$

Now from (28), equating the exponents $p_l+1$ and $3p_l-1$ gives

$$p_l+1=3p_l-1 \tag{29}$$

which leads to

$$p_l=1 \tag{30}$$

It needs to be noted that the same value of $p_l$ is obtained on equating the exponents $p_l+3$ and $3p_l+1$. Now, setting the coefficients of the linearly independent functions $\tanh^{p_l+j}\tau$, for $j=-1,1,3$, in (28), to zero gives

$$v=\alpha_l-2a_l\kappa_l-3\gamma_l\kappa_l^2-2\gamma_lB^2 \tag{31}$$

$$v=\alpha_l-2a_l\kappa_l-3\gamma_l\kappa_l^2-\left(3\lambda_l+2\nu_l\right)A_l^2-8\gamma_lB^2 \tag{32}$$

and

$$\left(3\lambda_l+2\nu_l\right)A_l^2+8\gamma_lB^2=0 \tag{33}$$

Now, it needs to be noted that on equating the two values of the velocities of the solitons from (31) and (32) also yields the same relation as given by (33). Moreover, from (28), the stand-alone linearly independent function is

$\tanh^{p_l-3}\tau$ whose coefficient must therefore vanish. This also leads to the same value of $p_l$ as in (30).

Again, from the real part equation given by (27), upon equating the exponent pairs $p_l+2p_{\bar{l}}$; $3p_l$ and $p_l+2$; $p_l+2p_{\bar{l}}$ also gives the same value of $p_l$ as in (30). Also, in (27) the stand alone linearly independent function $\tanh^{p_l-2}\tau$ must have its coefficient vanish which once again leads to (30). From (27), upon setting the coefficients of the linearly independent functions $\tanh^{p_l+j}\tau$ for $j=0,2$ implies

$$\omega_l=\alpha_l\kappa-l-a_l\left(\kappa_l^2+2B^2\right)-\gamma_l\kappa_l\left(\kappa_l^2+6B^2\right) \quad (34)$$

and

$$2\left(a_l+3\gamma_l\kappa_l\right)B^2+c_lA_{\bar{l}}^2+\left(b_l+\lambda_l\kappa_l\right)A_l^2=0 \quad (35)$$

Now equating the two values of the soliton velocity ($v$), for $l=1,2$ from (31) gives the free parameter $B$ of the soliton as

$$B=\left[\frac{\left(\alpha_1-\alpha_2\right)-2\left(a_1\kappa_1-a_2\kappa_2\right)-3\left(\gamma_1\kappa_1^2-\gamma_2\kappa_2^2\right)}{2(\gamma_1-\gamma_2)}\right]^{\frac{1}{2}} \quad (36)$$

which stays valid as long as

$$(\gamma_1-\gamma_2)\left\{\begin{matrix}\left(\alpha_1-\alpha_2\right)-2\left(a_1\kappa_1-a_2\kappa_2\right)\\-3\left(\gamma_1\kappa_1^2-\gamma_2\kappa_2^2\right)\end{matrix}\right\}>0 \quad (37)$$

From (33), for $l=1,2$, the two other free parameters are

$$A_l=\left[\frac{3\gamma_l\left\{\left(\alpha_{\bar{l}}-\alpha_l\right)+2\left(a_l\kappa_l-a_{\bar{l}}\kappa_{\bar{l}}\right)+3\left(\gamma_l\kappa_l^2-\gamma_{\bar{l}}\kappa_{\bar{l}}^2\right)\right\}}{\left(3\lambda_l+2\nu_l\right)\left(\gamma_l-\gamma_{\bar{l}}\right)}\right]^{\frac{1}{2}} \quad (38)$$

where $l=1,2$ and $\bar{l}=3-l$. This forces, after taking (37) into consideration, another constraint relation pair given by

$$\gamma_l\left(3\lambda_l+2\nu_l\right)<0 \quad (39)$$

Finally, equation (35) leads to another constraint relation that is given by

$$\frac{3\gamma_l\left(b_l+\lambda_l\kappa_l\right)}{3\lambda_l+2\nu_l}+\frac{c_l\gamma_{\bar{l}}}{3\lambda_{\bar{l}}+2\gamma_{\bar{l}}}=a_l+3\gamma_l\kappa_l \quad (40)$$

Hence, the dark 1-soliton solution in birefringent optical fibers in presence of strong Hamiltonian perturbation terms is given by (26) where the free parameters $A_l$ and $B$ are given by (38) and (36) respectively, while the velocities of the soliton are given by (31) or (32) and finally the wave numbers are given by (34). These lead to several constraint conditions that are seen in (37), (39) and (40).

The following figure is the profile of a dark soliton that is supported by the coupled NLSE given by (3) and (4) subjected to the aforementioned constraints.

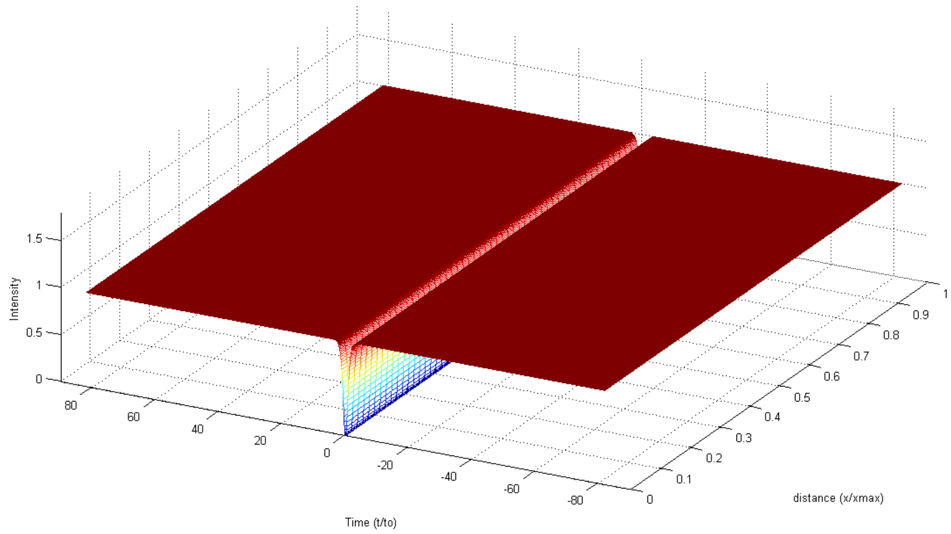


*Fig. 2. The typical shape of dark soliton.*

## 3. Conclusions

This paper studies the integrability aspects of the coupled NLSE that governs the propagation of solitons through birefringent optical fibers. Both bright and dark optical solitons are taken into consideration. There are several perturbation terms that are taken into consideration all of which are Hamiltonian type. Finally, there are the constraint conditions that fall out during the course of integrability. These conditions show that the solitons will exist only when these criteria remain valid. A couple of numerical simulations are also provided to illustrate the analytical development.

These results will be extended further in future when several other forms of nonlinearity will be studied. Additionally, the quasi-stationary soliton solutions will also be obtained, by the aid of multiple-scale perturbation analysis, in presence of both Hamiltonian as well as non-Hamiltonian type perturbation terms. Those results will be reported in future.

**Acknowledgement**

The first, fifth and sixth authors (AB, TH & OMA) would like to recognize and thankfully appreciate the support from King Saud University (KSU-VPP-117)

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