# Combo solitons of fifth-order nonlinear Schrodinger's hierarchy with complex-amplitude hypothesis

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We consider the evolution of attosecond light pulses in an optical fiber medium wherein the pulse propagation is governed by a fifth-order nonlinear Schrödinger equation with constant coefficients. In addition to the cubic nonlinearity and group velocity dispersion terms, the model incorporates the third-, fourth-, and fifth-order dispersion and nonlinear terms related to them. Using a complex envelope function ansatz, we find the analytical solitary wave solutions of the model under some parametric conditions. The reported solutions describe bright and dark solitary waves that propagate on a continuous wave background in the presence of higher-order effects. The constraint relations among the optical material parameters for the existence of these localized structures are also discussed.

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## 1. Introduction

Solitons are localized waves that can stably propagate over extremely large distances with neither attenuation nor change of shape, as their dispersion is exactly compensated by nonlinear effects. The study of these particle like objects has attracted considerable attention in recent years because they have been demonstrated in diverse area of science, such as plasma physics [1, 2], fluid dynamics [3], nonlinear optics [4, 5], Bose–Einstein condensates [6, 7], nuclear physics [8] and many others. One of the most important applications of solitons is in high-rate telecommunications with optical nonlinear fibers [9], where they are used as the carriers for the transmission of information.

From a theoretical point of view, the nonlinear Schrödinger (NLS) equation has been used successfully to describe the propagation of light pulses in nonlinear optical fibers and matter waves in Bose-Einstein condensates. The simplest form of this equation includes only basic effects on waves such as lowest-order dispersion and lowest-order nonlinearity. In the setting of optical fiber waveguides, the self-phase modulation (SPM) is the nonlinear effect due to the lowest dominant nonlinear susceptibility $\chi^{(3)}$ [10]. On the other hand, the dispersive properties of the light wave envelope are determined by the group velocity dispersion (GVD) [11]. Depending on the anomalous dispersion or the normal dispersion, the NLS equation allows for either bright or dark solitons, respectively [12].

With increasing light intensity, the nonlinear refractive index of a relatively large number of optical materials deviates from Kerr behavior [i.e., the refractive index varies linearly with the intensity $I$ of the light pulse as: $n = n_0 + n_2 I$, where $n_0$ is the linear refractive index coefficient, and $n_2$ is the nonlinear refractive index coefficient, which originates from the third-order susceptibility]. In this case, such materials exhibit not only third-order (or pure Kerr) nonlinearity but even fifth-order nonlinearity. Well known optical materials with nonideality of the nonlinear optical response include for example semiconductor waveguides (e.g., $Al_xGa_{1-x}As$, CdS, and $CdS_{1-x}Se_x$) and semiconductor-doped glasses (see, e.g., [13]). The dynamics of such systems should be described by the so-called cubic–quintic NLS model which represents one of the simplest extensions of the cubic model.

As optical pulses become shorter, many other higher-order effects such as third-order dispersion (TOD), self-steepening, and self-frequency shift become important in addition to the cubic and quintic terms [9]. Moreover, the effect of fourth-order dispersion (FOD) becomes significant on the propagation dynamics when the pulses are shorter than 10 femtoseconds [11]. Such dispersion may result in solitons with oscillating tails [14]. The behavior of wave packet in nonlinear media with these effects is governed by the higher-order nonlinear Schrödinger (HNLS) equation which includes the contribution of various physical phenomena on short-pulse propagation and generation. Depending on the practical situation, many versions of theoretical models for describing the wave dynamics have been proposed in the recent literature like the Hirota equation [17], the Sasa-Satsuma equation [18], and the Lakshmanan-Porsezian-Daniel (LPD) equation [19]. Recently, Ankiewicz *et al.* have introduced a novel NLS-type equation incorporating higher-order odd (third-order) and even (fourth-order) terms and presented its soliton solutions and approximate rogue wave solutions [20]. Subsequently, the existence of various shapes of soliton structures governed by this extended model has been extensively discussed. For instance, Bendahmane et *al*. have discussed this HNLS equation under nonvanishing boundary conditions by employing the ansatz method [22]. As an interesting result, they obtained a W-shaped solitary wave solution with a platform underneath in addition to envelope solutions of the bright and dark types. In Ref. [20], dipole soliton solutions of such extended model have also been derived by adopting a complex amplitude ansatz that is composed of the product of bright and dark solitary waves.

A natural extension of the theoretical studies of nonlinear structures in dynamical systems is to include the effect of the fifth-order terms besides third- and fourth-order terms. Very recently, a fifth-order nonlinear equation in the NLS hierarchy which contains fifth-order dispersion and nonlinear terms related to it was presented [23]. Remarkably, this new envelope equation adds an extra new quintic operator (beginning with fifth-order dispersion) with respect to the extended model with cubic and quartic terms given in Ref. [20]. It is worth mentioning that the inclusion of fifth-order terms in the governing equation is essential to study the attosecond pulse propagation in nonlinear media. In the framework of this model, the basic one-soliton solution as well as the second- and third-order soliton solutions have been successfully derived by using of the Darboux method [23].

In terms of practical applications, propagations of ultrashort (femtosecond) pulses are of particular interest because of their wide applications in many different areas such as ultrahigh-bit-rate optical communication systems, ultrafast physical processes, infrared time-resolved spectroscopy, and optical sampling systems [24]. During recent years, attosecond pulses have also been the subject of extensive research in nonlinear fiber media. Therefore, a study of solitons or solitary waves in the femtosecond and attosecond regimes is significant especially for the recent fiber technology.

In the present work, we derive analytical combined solitary wave solutions of a family of the fifth-order equation of the NLS hierarchy describing the attosecond pulse propagation in an optical fiber medium. In the used governing equation, we have considered arbitrary real parameters $r_j$ (with $j=1,..,13$) in front of every term of the newly model introduced in Ref. [23]. This allows us to examine the individual influence of each effect on the propagation properties of the existing localized structures. Note that having an explicit analytic solution of a model with arbitrary coefficients has the advantage that we can also consider all particular cases analytically. It should be noted that the considered model collapses to the regular equation introduced [23] if setting the dependent parameters equal to 1. Here, we will adopt the ansatz solution of Li et *al.* [25] to find the combined solitary wave solutions under some parametric conditions.

The rest of this paper is structured as follows. In Sec. II, we introduce the theoretical model and give its particular cases. In Sec. III, we present two different solitary wave solutions of the fifth-order equation of the NLS hierarchy and discuss their characteristic and formation conditions of their existence. In Sec. IV, we give some concluding remarks and perspectives.

## 2. Governing model

Under investigation in this paper is the following fifth-order equation of the NLS hierarchy equation with constant coefficients:

$$
\begin{aligned}
& iE_x + \frac{E_{tt}}{2} + |E|^2 E + \gamma\ (E_{tttt} + 6r_1 E_t^2 E^* + 4r_2 |E_t|^2 E + 8r_3 |E|^2 E_{tt} + \\
& 2r_4 E_{tt}^* E^2 + 6r_5 |E|^4 E)\ \text{-}i\alpha_3\ (E_{ttt} + 6r_6 |E|^2 E_t + 6r_7 E^2 E_t^*) \\
& \text{-}i\delta\ (E_{ttttt} + 10r_8 |E|^2 E_{ttt} + 30r_9 |E|^4 E_t + \\
& 10r_{10} E E_t E_{tt}^* + 10r_{11} E E_t^* E_{tt} + 20r_{12} E^* E_t E_{tt} + 10r_{13} E_t^2 E_t^*) = 0,
\end{aligned} \tag{1}
$$

which can be reduced into the regular equation introduced in Ref. [23] when $r_j = 1$ with $j = 1,..,13$. Here $E(x,t)$ is the complex envelope of the wave, *x* is the propagation variable and *t* is the transverse variable (time in a moving frame). Also, the coefficients $\alpha_3$, $\gamma$ and $\delta$ are the third-, fourth- and fifth-order dispersion parameters, respectively. In our study, the constants $r_j$ (with $j=1,..,13$ ) are free real parameters.

The model (1) contains many special NLS-type equations, such as the cubic NLS equation for the case $\alpha_3 = \gamma = \delta = 0$ [26], the Hirota equation for the case $\gamma = \delta = 0$, $r_6 = 1$ and $r_7 = 0$ [17], the Sasa-Satsuma equation in the case $\gamma = \delta = 0$, $r_6 = 3/2$ and $r_7 = 1/2$ [18], the LPD equation for the case $\alpha_3 = \delta = 0$ and $r_i = 1$ (with $i = 1,..,5$) [21], and the extended NLS equation with third- and fourth-order terms for the case $\delta = 0$ [22].

It is of interest to determine the solitary wave solutions of Eq. (1). Obtaining exact soliton structures of this nonlinear equation is important from both theoretical and practical point of views. It is worth mentioning that exact solutions when they exist can help one to calculate certain important physical quantities analytically as well as serving as diagnostics for simulations [29]. In what follows, we will adopt the complex ansatz solution of Li et *al.* [25] to find exact combined solitary wave solutions of the considered model under certain parametric conditions. Such ansatz has been successfully applied to solve many higher-order NLS models [26-35]).

## 3. The method of solution

To start with, we search for the solutions of the physical field $E(x,t)$ with amplitude $A(x,t)$ and linear phase shift $\varphi(x,t)=kx\text{-}\Omega t$ as [25]

$$E(x,t)=A(x,t)\exp\left[i\varphi(x,t)\right], \tag{2}$$

where $k$ and $\Omega$ are real parameters describing wave number and frequency shift, respectively. Inserting Eq. (2) in Eq. (1) and removing the exponential term, we write the resulting equation as

$$\begin{aligned}&iA_x+ia_1A_t+a_2A_{tt}+a_3|A|^2A\text{-}ia_4|A|^2A_t\text{-}ia_5A^2A_t^*\text{-}a_6A+\\&a_7|A|^4A+a_8A_{tttt}\text{-ia}_9A_{ttt}+a_{10}A_t^2A^*+a_{11}|A_t|^2A+\\&a_{12}|A|^2A_{tt}+a_{13}A^2A_{tt}^*\text{-}i\delta A_{ttttt}\text{-}ia_{14}|A|^2A_{ttt}\text{-}ia_{15}|A|^4A_t\\&\text{-}ia_{16}AA_tA_{tt}^*\text{-}ia_{17}AA_t^*A_{tt}\text{-}ia_{18}A^*A_tA_{tt}\text{-}ia_{19}|A_t|^2A_t=0,\end{aligned} \tag{3}$$

where we have introduced the parameters $a_n$ (with $n=1,...,19$):

$$a_1=\text{-}\Omega+3\alpha_3\Omega^2+4\gamma\Omega^3\text{-}5\delta\Omega^4,$$

$$a_2=\frac{1}{2}\text{-}3\alpha_3\Omega\text{-}6\gamma\Omega^2+10\delta\Omega^3,$$

$$a_3=1\text{-}6\alpha_3\Omega(r_6\text{-}r_7)\text{-}2\gamma\Omega^2(3r_1\text{-}2r_2+4r_3+r_4)$$
$$+10\delta\Omega^3(r_8+r_{10}\text{-}r_{11}+2r_{12}\text{-}r_{13}),$$

$$a_4=6\alpha_3r_6+4\gamma\Omega(3r_1\text{-}r_2+4r_3)\text{-}$$
$$10\delta\Omega^2(3r_8+r_{10}\text{-}2r_{11}+6r_{12}\text{-}2r_{13}),$$

$$a_5=6\alpha_3r_7+4\gamma\Omega(r_2\text{-}r_4)\text{-}10\delta\Omega^2(-2r_{10}+r_{11}+r_{13}),$$

$$a_6=k+\frac{1}{2}\Omega^2\text{-}\alpha_3\Omega^3\text{-}\gamma\Omega^4+\delta\Omega^5,$$

$$a_7=6(\gamma r_5\text{-}5r_9\delta\Omega),$$

$$a_8=\gamma\text{-}5\delta\Omega,$$

$$a_9=\alpha_3+4\gamma\Omega\text{-}10\delta\Omega^2,$$

$$a_{10}=6r_1\gamma\text{-}10\delta\Omega(4r_{12}\text{-}r_{13}),$$

$$a_{11}=4r_2\gamma+20\delta\Omega(r_{10}\text{-}r_{11}\text{-}r_{13}),$$

$$a_{12}=8\gamma r_3+10\delta(r_{11}\text{-}3r_8\text{-}2r_{12}),$$

$$a_{13}=2r_4\gamma\text{-}10r_{10}\delta,$$

$$a_{14}=10r_8\delta,$$

$$a_{15}=30r_9\delta,$$

$$a_{16}=10r_{10}\delta,$$

$$a_{17}=10r_{11}\delta,$$

$$a_{18}=20r_{12}\delta,$$

and

$a_{19}=10r_{13}\delta$.

By adopting the complex amplitude ansatz introduced by Li et *al.* [25]:

$$A(x,t)=i\beta+\lambda\tanh\left[\eta(t\text{-}\chi x)\right]+i\rho\,\text{sech}\left[\eta(t\text{-}\chi x)\right], \tag{4}$$

where $\eta$ and $\chi$ are the pulse width and shift of inverse group velocity, respectively. Also, $\lambda$ and $\rho$ represent the single dark and bright soliton amplitude, respectively. From the ansatz solution (4), one can see that when the time variable approaches infinity, the amplitude of solitary wave solutions is nonzero. It should be noted that $\eta$, $\chi$, $k$ and $\Omega$ are all real values but $\beta$, $\lambda$ and $\rho$ can be real or complex numbers depending on the equation parameters [32]. Accordingly, the solitary wave amplitude $A(x,t)$ can be written as:

$$|A(x,t)|=\left\{\begin{matrix}\lambda^2+\beta^2+2\beta\rho\,\text{sech}\left[\eta(t\text{-}\chi x)\right]+\\(\rho^2\text{-}\lambda^2)\text{sech}^2\left[\eta(t\text{-}\chi x)\right]\end{matrix}\right\}^{1/2}, \tag{5}$$

and its corresponding nonlinear phase shift $\varphi_{NL}(x,t)$ is of the form:

$$\varphi_{NL}(x,t)=\arctan\left(\frac{\beta+\rho\,\text{sech}\left[\eta(t\text{-}\chi x)\right]}{\lambda\tanh\left[\eta(t\text{-}\chi x)\right]}\right). \tag{6}$$

It is interesting to note that if $\beta=\rho=0$, and then the ansatz solution (4) reduces to an original dark soliton. When $\beta=\lambda=0$, one recovers a bright-type soliton. One can also point out that presence of the parameters $\beta$, $\lambda$ and $\rho$ permits the ansatz (4) to describe a combined solitary wave solution [25].

Now, inserting (4) into (3), expanding tanh terms to sech terms, and equating the coefficients for the independent terms equal to zero, we obtain the following 13 independent parametric equations:

$$\beta\left[a_3(\beta^2+\lambda^2)\text{-}a_6+a_7(\beta^2+\lambda^2)^2\right]=0, \tag{7a}$$

$$\lambda\left[a_3(\beta^2+\lambda^2)\text{-}a_6+a_7(\beta^2+\lambda^2)^2\right]=0, \tag{7b}$$

$$\rho\eta^2\left[a_2+a_{12}(\beta^2+\lambda^2)+a_{13}(\beta^2-\lambda^2)+a_8\eta^2\right] +\rho\begin{bmatrix}a_3(3\beta^2+\lambda^2)+2a_5\beta\lambda\eta-a_6+\\ a_7(5\beta^2+\lambda^2)(\beta^2+\lambda^2)\end{bmatrix}=0, \tag{7c}$$

$$\lambda\eta\begin{bmatrix}-\chi+a_1-a_4(\beta^2+\lambda^2)+a_5(2\rho^2+\beta^2-\lambda^2)\\ -a_{15}(\beta^2+\lambda^2)^2\end{bmatrix} -\lambda\eta^3\begin{bmatrix}4a_9+4a_{14}(\beta^2+\lambda^2)+16\delta\eta^2\\ -(a_{16}+a_{17}-a_{18})\rho^2\end{bmatrix} +\beta\begin{bmatrix}a_3(3\rho^2-\lambda^2)+2a_7\beta^2(5\rho^2-\lambda^2)+\\ 2a_7\lambda^2(3\rho^2-\lambda^2)\end{bmatrix} +\beta\rho^2\eta^2(a_{10}+a_{11}+2a_{12})+2a_{13}\beta\eta^2(\rho^2-2\lambda^2)=0, \tag{7d}$$

$$\rho\eta^2\begin{bmatrix}-2a_2-20a_8\eta^2+a_{10}(\rho^2-2\lambda^2)+a_{11}\rho^2+\\ a_{12}(\rho^2-2\beta^2-3\lambda^2)+a_{13}(\rho^2-2\beta^2-\lambda^2)\end{bmatrix} -\beta\rho\lambda\eta^3\,(8a_{14}-a_{16}+a_{17}+3a_{18})- 2\beta\rho\lambda\eta\left[a_4+2a_{15}(\beta^2+\lambda^2)\right] +\rho\begin{bmatrix}a_3(\rho^2-\lambda^2)+2a_7\beta^2(5\rho^2-3\lambda^2)+\\ 2a_7\lambda^2(\rho^2-\lambda^2)\end{bmatrix}=0, \tag{7e}$$

$$\lambda\eta^3\begin{bmatrix}6a_9+120\delta\eta^2-2a_{14}(2\rho^2-3\beta^2-5\lambda^2)-\\ 2a_{16}(\rho^2-\lambda^2)-2a_{17}(2\rho^2-\lambda^2)+2a_{18}\lambda^2\\ -a_{19}\rho^2\end{bmatrix}-\lambda\eta\begin{bmatrix}(\rho^2-\lambda^2)(a_4+a_5+2a_{15}\lambda^2)+\\ 2a_{15}\beta^2(3\rho^2-\lambda^2)\end{bmatrix}+ \beta a_7(\rho^2-\lambda^2)(5\rho^2-\lambda^2) -\beta\eta^2\begin{bmatrix}a_{10}(\rho^2+\lambda^2)+a_{11}(\rho^2-\lambda^2)+\\ 4a_{12}\rho^2+4a_{13}(\rho^2-\lambda^2)\end{bmatrix}=0, \tag{7f}$$

$$\rho\eta^2\left[24a_8\eta^2-(\rho^2-\lambda^2)(a_{10}+a_{11}+2a_{12}+2a_{13})\right]+ 4\beta\rho\lambda\eta^3\,(3a_{14}+a_{18}) -4a_{15}\beta\rho\lambda\eta\,(\rho^2-\lambda^2)+a_7\rho(\rho^2-\lambda^2)^2=0, \tag{7g}$$

$$\lambda\eta^3\left[(\rho^2-\lambda^2)(6a_{14}+2a_{16}+2a_{17}+2a_{18}+a_{19})\right]- \lambda\eta\left[120\delta\eta^4+a_{15}(\rho^2-\lambda^2)^2\right]=0, \tag{7h}$$

$$\rho\eta\begin{bmatrix}-\chi+a_1+a_5(\lambda^2-\beta^2)-a_4(\beta^2+\lambda^2)\\ -a_{15}(\beta^2+\lambda^2)^2\end{bmatrix} -\rho\eta^3\left[a_9+a_{14}(\beta^2+\lambda^2)+\delta\eta^2\right]+ 2\beta\lambda\rho\left[a_3+2a_7(\beta^2+\lambda^2)+a_{13}\eta^2\right]=0, \tag{7i}$$

$$\lambda\eta^2\begin{bmatrix}-2a_2-8a_8\eta^2-(a_{10}-a_{11})\rho^2-2a_{12}(\beta^2+\lambda^2)+\\ 2a_{13}(\rho^2+\beta^2-\lambda^2)\end{bmatrix} -\beta\rho^2\eta^3\,(2a_{14}+a_{16}+a_{17}+a_{18})-2a_4\beta\rho^2\eta- 4a_{15}\beta\rho^2\eta(\beta^2+\lambda^2) +\lambda\begin{bmatrix}a_3(\rho^2-\lambda^2)+2a_7\beta^2(3\rho^2-\lambda^2)+\\ 2a_7\lambda^2(\rho^2-\lambda^2)\end{bmatrix}-2a_5\beta\eta(\rho^2-\lambda^2)=0, \tag{7j}$$

$$\rho\eta\begin{bmatrix}(\rho^2-\lambda^2)(-a_4-a_5)-2a_{15}\beta^2(3\rho^2-\lambda^2)-\\ 2a_{15}\lambda^2(\rho^2-\lambda^2)\end{bmatrix} +2\beta\lambda\rho(2a_7(\rho^2-\lambda^2)-(a_{10}+2a_{12})\eta^2)+ \rho\eta^3\begin{bmatrix}6a_9-a_{14}(\rho^2-6\beta^2-7\lambda^2)\\ -(\rho^2-\lambda^2)(a_{16}+a_{17})-a_{18}(\rho^2-3\lambda^2)-\\ a_{19}\rho^2+60\delta\eta^2\end{bmatrix}=0, \tag{7k}$$

$$\lambda\eta^2\left[24a_8\eta^2-(\rho^2-\lambda^2)(a_{10}+a_{11}+2a_{12}+2a_{13})\right] +2\beta\eta^3\left[6a_{14}\rho^2+(a_{16}+a_{17})(\rho^2-\lambda^2)+a_{18}(\rho^2+\lambda^2)\right] +a_7\lambda(\rho^2-\lambda^2)^2-4a_{15}\beta\rho^2\eta(\rho^2-\lambda^2)=0, \tag{7l}$$

$$\rho\eta^3\left[(\rho^2-\lambda^2)(6a_{14}+2a_{16}+2a_{17}+2a_{18}+a_{19})\right]- \rho\eta\left[120\delta\eta^4+a_{15}(\rho^2-\lambda^2)^2\right]=0. \tag{7m}$$

## 4. Results and discussion

To obtain solitary wave solutions for the model (1), we need to impose some restrictions on the dependent parameters so that Eqs. (7a) - (7m) become compatible. Here we have found two different types of solitary wave solutions of Eq. (1) under certain parametric conditions.

### 4.1. Bright solitary wave solution

The first solitary wave solution we obtained here is of the form

$$A(x,t)=i\beta\mp\sqrt{2}\beta\,\mathrm{sech}\left[\eta(t-\chi x)\right], \tag{8}$$

for the following conditions:

$$a_4+a_5=0,\ \ a_{10}+a_{11}=0,\quad a_3=a_9=a_{12}=a_{13}=a_{14}=0,\ \ a_2=a_8=a_7=0, \tag{9}$$

implying, in this case, that $\lambda=0$ and $\rho=\pm i\sqrt{2}\beta$ in the ansatz solution (4). Substitution of these results in Eqs. (7a) - (7m) yields the following solitary wave parameters:

$$\eta^2 = -\frac{30r_9}{r_{13}}\beta^2, \tag{10}$$

$$\Omega = \frac{\gamma}{5\delta}, \tag{11}$$

$$\chi = -\Omega + 3\alpha_3\,\Omega^2 + 4\gamma\,\Omega^3 - 5\delta\,\Omega^4 - 30r_9\delta\beta^4 - \delta\eta^4, \tag{12}$$

$$k = -\frac{1}{2}\Omega^2 + \alpha_3\,\Omega^3 + \gamma\,\Omega^4 - \delta\,\Omega^5, \tag{13}$$

together with the parametric conditions:

$$\begin{aligned}
&r_{13} = \frac{5}{2}(r_{10} + r_{11} + 2r_{12}), \quad r_8 = 0,\\
&\frac{\gamma}{\delta} = \frac{5r_{10}}{r_4}, \quad 4r_3 r_{10} = r_4(2r_{12} - r_{11}),\\
&3r_1 - r_{13} = 2(r_{11} + 2r_{12} - r_2 - r_{10}), \quad r_5 = r_9, \quad \alpha_3 = \frac{-2\gamma^2}{5\delta},\\
&2\,\gamma r_{10}^2\begin{bmatrix}6(r_7 - r_6) + \\ (2r_2 + 3r_1 - 4r_3 - r_4 - r_{10} + r_{11} - 2r_{12} + r_{13})\end{bmatrix} - r_4^2 = 0,\\
&\delta^2 < \frac{-9\gamma^3}{50},\\
&6(r_7 + r_6) - 2(3r_1 - r_4 + 4r_3 - r_{10} - 3r_{12}) - r_{11} - r_{13} = 0.
\end{aligned} \tag{14}$$

Therefore, the exact solitary wave solution of Eq. (1) can be written as:

$$E(x,t) = \left\{i\beta \mp \sqrt{2}\beta\,\mathrm{sech}\left[\eta(t - \chi x)\right]\right\}e^{i(kx - \Omega t)}, \tag{15}$$

and its intensity is given by

$$\left|E(x,t)\right|^2 = \beta^2\left(1 + 2\,\mathrm{sech}^2\left[\eta(t - \chi x)\right]\right). \tag{16}$$

Physically, Eq. (15) describes a bright pulse that propagates on a continuous wave background which originates from the higher-order effects. One can see from Eqs. (11), (12) and (13) that the solitary wave parameters are dependent on the coefficients $\alpha_3, \gamma$ and $\delta$ that control independently the values of third-order dispersion $E_{ttt}$ , fourth-order dispersion $E_{tttt}$ , and that of fifth-order dispersion $E_{ttttt}$ . Moreover, Eq. (10) shows that one must require $r_9 r_{13} < 0$ for obtaining a real value of the pulse width $\eta$ .

### 4.2. Dark solitary wave solution

We also find a second solitary wave solution for Eq. (1) of the form:

$$A(x,t) = i\beta + \lambda\tanh\left[\eta(t - \chi x)\right], \tag{17}$$

under the following necessary conditions:

$$\begin{aligned}
&a_4 + a_5 = 0,\ a_{10} + a_{11} = 0,\ a_2 = a_3 = a_7 = a_9 = a_{14} = a_{19} = 0,\\
&a_{12} - a_{13} = 0,\\
&a_{10} - 2a_{13} = 0,\ a_{16} + a_{17} - a_{18} = 0.
\end{aligned} \tag{18}$$

showing, in this case, that $\rho = 0$ in the ansatz solution (4). By inserting these results in Eq. (7a) - (7m), one can obtain the solitary wave parameters as

$$\eta^3 = \frac{\gamma^2 r_5[2r_9(r_2 - r_4) - r_5(r_{11} - 2r_{10})] + 15\alpha_3 r_7 r_9^2\delta}{20r_9\delta\gamma(r_9 - r_5)}\beta\lambda, \tag{19}$$

$$\Omega = \frac{\gamma\, r_5}{5\delta\ r_9}, \tag{20}$$

$$k = -\frac{1}{2}\Omega^2 + \alpha_3\,\Omega^3 + \gamma\,\Omega^4 - \delta\,\Omega^5, \tag{21}$$

$$\begin{aligned}
&\chi = -\Omega + 3\alpha_3\,\Omega^2 + 4\gamma\,\Omega^3 - 5\delta\,\Omega^4 - 16\delta\eta^4\\
&-30r_9\delta(\lambda^2 + \beta^2)^2 +\\
&+[6\alpha_3\, r_7 + 4\gamma\,\Omega(r_2 - r_4) + 10\delta\Omega^2(2r_{10} - r_{11})]\beta^2 -\\
&4(2r_2\gamma - 10r_{10}\gamma)\beta\lambda\eta,
\end{aligned} \tag{22}$$

with following constraint relations:

$$\begin{aligned}
&r_8 = r_{13} = 0,\ r_3 = r_4,\quad 2r_{12} = r_{10} + r_{11},\\
&5r_9^2\alpha_3\delta = \gamma^2 r_5(2r_5 - 4r_9),\quad r_9(3r_1 + 2r_2) = 4r_5 r_{11},\\
&3(r_6 + r_7)(r_5 - 2r_9) = 3r_9(r_1 + r_3) - 2r_5 r_{12},\\
&r_9 - 6\alpha_3 r_9\Omega - 4\gamma\Omega^2(3r_9 - 4r_5) = 0,\\
&r_9 - 6\alpha_3 r_9\Omega(r_6 - r_7) + 2\gamma\Omega^2[2r_5 r_{10} - r_9(3r_1 - 2r_2 + 5r_3)] = 0,
\end{aligned} \tag{23}$$

Having obtained the pulse parameters $\eta$, $\Omega$, k and $\chi$ , we can now write the analytic solitary wave solution of Eq. (1) as follows

$$E(x,t) = \left\{i\beta + \lambda\tanh\left[\eta(t - \chi x)\right]\right\}e^{i(kx - \Omega t)}, \tag{24}$$

and its intensity is given by

$$\left|A(x,t)\right|^2 = \beta^2 + \lambda^2 - \lambda^2\,\mathrm{sech}^2\left[\eta(t - \chi x)\right]. \tag{25}$$

Equation (24) describes a dark like solitary wave with a width related to the product of $\beta\ \lambda$ . From the results obtained above, one should note the universal influence of the fifth-order dispersion parameter on the solitary wave properties. It influences the form of the width, wave number, frequency shift, and the shift of inverse group velocity of the propagating solitary pulses.

## 5. Conclusion

In this work, we studied the fifth-order equation of the nonlinear Schrödinger hierarchy, which governs the propagation of attosecond light pulses in an optical fiber. By adopting a complex envelope function ansatz, we have found the bright and gray solitary pulse (a dark pulse with nonzero minimum in intensity) solutions for the model on a continuous-wave background. The constraint relations among the parameters for the existence of these localized structures are also given. These attosecond pulses are helpful to increase the capacity of carrying information in order to make ultra-fast communication.

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