Completing the Lotmar model for the human eye with the crystalline lens refraction index variation function

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In the scientific papers are presented various models for the optical system of the human eye, considered ideal, known by the term of emmetropic eye. Little information can be found regarding the extra-paraxial aberrations of the human eye, like: Spot diagram, transversal aberrations, distribution of light in the image point, aberration of the wavefront and modulation transfer function. This paper presents a synthesis of the most important optical aberrations of the human eye, more precisely the aberration of the wavefront and modulation transfer function (MTF) and a method for the calculation of the refractive index of the crystalline lens, which will lead to ideal aberrations. The mathematical model selected is based on the spherical diopters hypothesis and is limited by diffraction.

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Keywords: Eyeball, Refractive index, Crystalline lens, Interpolation, Wavefront aberration, Modulation transfer function (MTF)

1. Introduction

In the scientific papers are presented various mathematical models of the optical system of the emmetropic eye (Lotmar, Gullstrand, LeGrand, etc.).

Little information can be found regarding the extra-paraxial aberrations, like: Spot diagram, transversal aberrations, distribution of light in the image point, aberration of the wavefront and modulation transfer function.

This paper presents an evaluation of the wavefront aberration and modulation transfer function (MTF) for a selected model of the human eye. The use of the calculation formulas for these aberrations leads to very high values as compared to the reality. This was to be expected, considering that the refractive index is constant within the mass of the lens. In reality, we know that the refractive index of the lens is variable throughout its mass by a law, which unfortunately, is not known.

The paper presents a method for calculating the refractive index of the lens, which will lead to ideal aberrations, meaning the wavefront aberration with values very close to zero and the MTF overlapped on the ideal curve.

By knowing the manner in which the refractive index of the lens changes, the selected human eye achieves paraxial and extra-axial harmony. The paraxial and extra-axial harmony allows the mathematical simulation of the subjective evaluation, the correction using eyeglass lenses or contact lenses.

The mathematical model selected is based on the spherical diopters hypothesis, hypothesis accepted for all ideal models. It has geometrical aberrations equal to zero and the resolution limited by the diffraction phenomenon. Based on this statement we can say that the variation pattern for the refractive index of the lens creates an optical system at the diffraction limit. In order to process the data and achieve their graphical representation was employed an optical calculation software, designed by the authors based on the optical calculation formulas, in an extension of the Pascal software, named Delphi 2006.

2. Presenting the mathematical model selected for the human eye [1-5]

In order to create a mathematical model of the human eyeball, the information collected from the scientific papers must be processed in a uniform manner, to obtain the calculation formulas, just like in the technical optics. From the analysis conducted on various models of the human eye, studied from the scientific papers, were obtained the dimensional limits for the ocular environments.

In table 1 are summarized the statistical data for enclosing of the human eye parameters, and in table 2 are presented the values of the refractive indexes for the components of the human eye, excerpted from the work [1].

Table 1. Statistical enclosing of the parameters of the optical system of the human eye.

Name	Interval
1. Cornea thickness	0.40 . . 0.60 mm
2. Cornea anterior radius	7.00 . . 8.65 mm
3. Cornea posterior radius	6.20 . . 6.80 mm
4. Anterior chamber depth	2.80 . . 4.60 mm
5. Lens anterior radius	8.80 . . 11.90 mm
6. Lens posterior radius	-5.60 . . - 6.00 mm
7. Lens thickness	3.60 . . 4.50 mm
8. Lens power	15.00 . . 27.00 dpt
9. Axial length	20.00 . . 29.5 mm
10. Ocular power	54.00 . . 65.00 dpt
11. Eyeball radius	11.00 . . 12.50 mm
12. Rotation center for the eyeball	12.80 . . 13.50 mm
13. Iris diameter	2.00 . . 8.00 mm
14. Iris abscissa	2.10 . . 2.25 mm

Table 2. Refractive indexes for the base radiations for optical components of the eyeball.

		n_C λ=656.2725 nm	n_D λ=589.2937 nm	n_F λ=468.1327 nm	n_g λ=435.8343 nm	ν_D λ=587.2937 nm
1	**Cornea**	1.3751	1.3771	1.3818	1.3857	56.2835
2	**Aqueous humor**	1.3354	1.3374	1.3418	1.3454	52.7187
3	**Lens**	1.4175	1.4200	1.4254	1.4307	53.4645
4	**Vitreous humor**	1.3341	1.3360	1.3404	1.3440	53.333

By studying the various models of the human eye, the Lotmar model was determined to be the most convenient for the proposed demonstrations. The construction of the selected ocular model, with information for dimensioning and the characterization of its environments, is presented in Fig. 1. For the calculation of the aberrations of the human eye must be determined the refractive indexes for at least 4 wavelengths**.** The refractive index will be obtained for any wavelength by using the interpolation polynomial given by the Laurent series:

$$n^2(\lambda) = \sum_{j=0}^{\kappa} \mathrm{A}_j \lambda^{2j} + \sum_{j=1}^{\Lambda} \mathrm{A}_{j+\kappa} \lambda^{-2j} \qquad (1)$$

The current version used for the visible domain considers k = 1 and L = 4, resulting:

$$n^2 = A_0 + A_1\lambda^2 + A_2\lambda^{-2} + A_3\lambda^{-4} + A_4\lambda^{-6} + A_5\lambda^{-8} \qquad (2)$$

The determination of the unknown variables, meaning the coefficients from formula (2), will be made by using the method of the least squares. The formulas were obtained by assuming that the rate of the curve of the refractive index by the wavelength of the components of the optical system of the eyeball is similar to the one for the optical glasses. With the aid of the proposed optical calculation software is obtained the graph of the values of the refractive index from Fig. 2.

In the same manner will be processed the values for the other components of the human eye, the data obtained being presented in Table 3.

In order to obtain accurate values, the values of the coefficients are used with the maximum number of decimals. Due to technical considerations the results are presented with just 4 decimals. In table 3 are presented the coefficients of the interpolation polynomials for the determination of the variation graphs for the refractive indexes for all ocular environments.

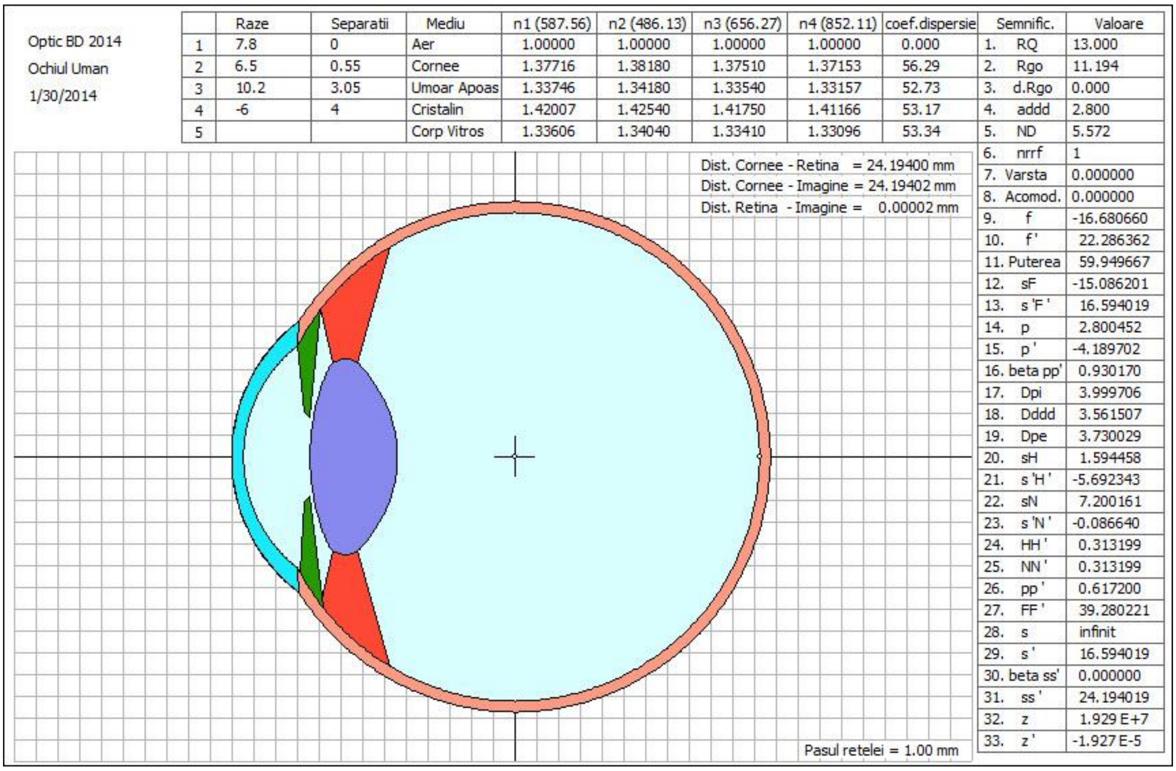

Fig. 1. Mathematical model selected for the human eye

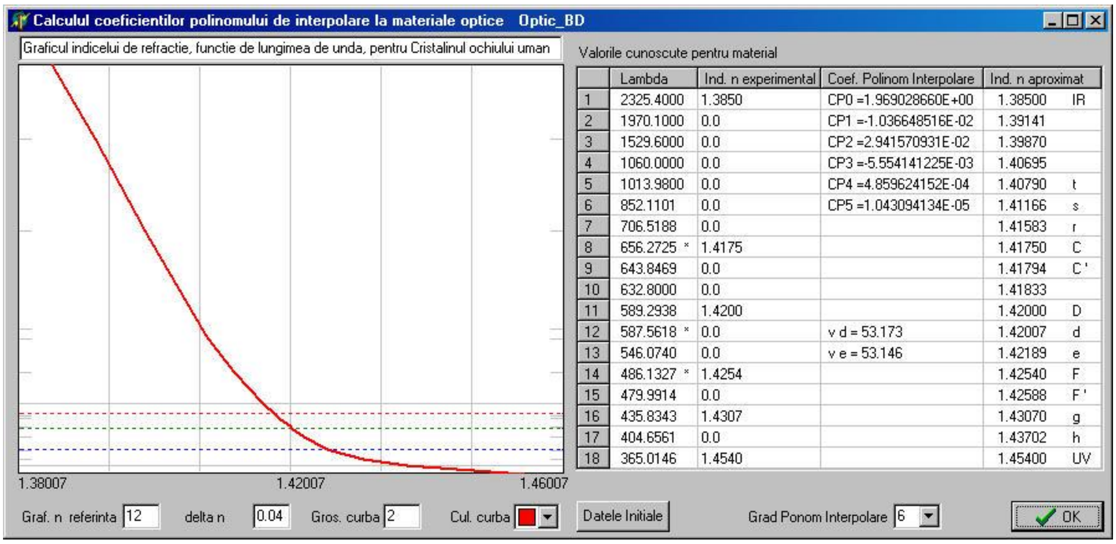

Fig. 2. Determination of the coefficients of the interpolation polynomial for the human eye lens

Table 3. Coefficients of the interpolation polynomial for the refractive indexes for the environments of the eyeball

	Cornea	**Aqueous humor**	**Lens**	**Vitreous humor**
A_0	1.8755 E+00	1.7636 E+00	1.9690 E+00	1.7694 E+00
A_1	-4.9532 E-03	-4.2019 E-03	-1.0366 E-02	-2.9923 E-03
A_2	3.9953 E-03	7.7294 E-03	2.9415 E-02	-2.5009 E-03
A_3	2.6310 E-03	1.5954 E-03	-5.5541 E-03	5.4257 E-03
A_4	-5.9387 E-04	-5.2328 E-04	4.8596 E-04	-1.1432 E-03
A_5	4.9787 E-05	5.0380 E-05	1.0430 E-05	8.7305 E-05

When we remove the selected model from the natural emmetropia state, due to the variation within the body of the refractive index of the lens, it becomes ametropic, the correction involves the positioning in front of the eye of a corrective lens. In order to apply the design methods for the optical systems must also be known the coefficients of the interpolation polynomial for the material composing the corrective lens. The material selected for the corrective lens is presented in table 4.

Table 4. Refractive indexes for the material composing the contact lens.

Material	n_F λ=468.1327 nm	n_D λ=589.293 nm	n_C λ=656.272 nm	ν_D λ=587.2937 nm
Methyl methacrylate styrene copolymer (NAS)	1.574	1.563	1.558	33.5

With the aid of the calculation software are obtained the coefficients of the interpolation polynomial for the refractive index for the material that could be used for manufacturing the corrective lens. They are presented in the Table 5.

The graph of the interpolation polynomial resulted for the selected material is presented in Fig. 3.

Table 5. Coefficients of the interpolation polynomial for the refractive index for the materials that could be used for manufacturing the contact lens.

	NAS
A_0	2.3365 E+00
A_1	1.5363 E-02
A_2	6.5108 E-02
A_3	1.3964 E-02
A_4	1.9054 E-03
A_5	7.8811 E-05

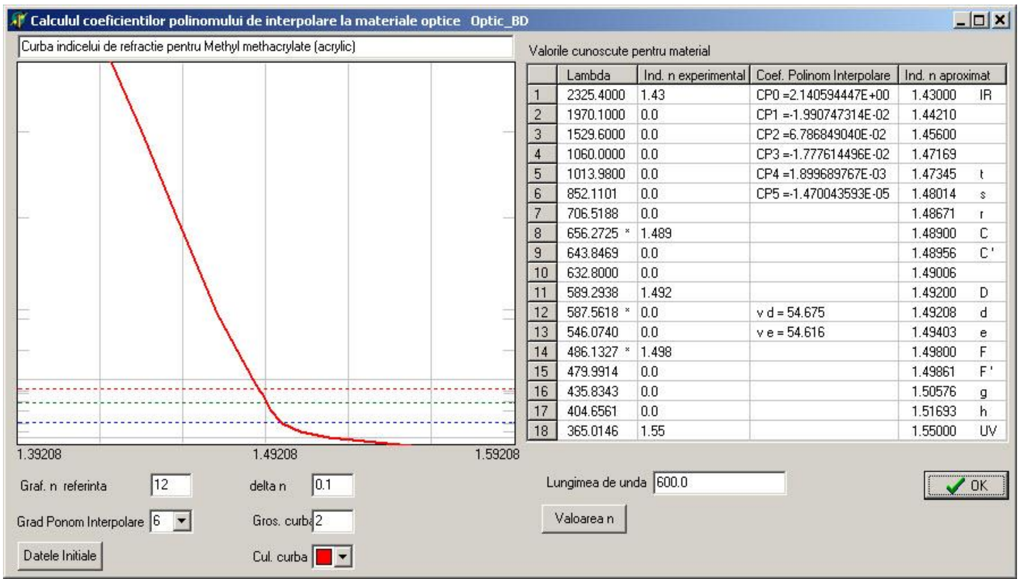

Fig. 3. Curve of the coefficients of the interpolation polynomial for the material Methyl methacrylate styrene copolymer

3. Aberrations of the optical system of the eyeball [2-7]

In order to calculate the extra-paraxial aberrations, is constructed a beam of optical rays leaving from the object point, having a uniform distribution on the surface of the entry pupil. For the passing of the rays through the optical system are employed the complex formulas of 3D tracing. With the aid of the constructed beam of optical rays, can be conducted the analysis of a set of aberrations, like: Spot diagram, transversal aberration, distribution of light in the image point, aberration of the wavefront and modulation transfer function.

3.1. Wavefront aberration

The wavefront aberration represents the most important information that can describe the state of correction for an optical subassembly. By definition the wavefront aberration is given by formula 3.

$$W(x,y)=\sum_j n_j\left(\overline{D}_j-D_j\right) \tag{3}$$

In this formula $\overline{D}_j$ represents the basic geometric pathway of the main pupil radius; D_j represents the basic optical pathway of a given radius, and n_j represents the refractive index of the environments traversed by the optical rays.

We approximate the wavefront aberration with the aid of the Malacara polynomial presented in formula 4. [6]:

$$W(x,y)=\sum_{i=0}^{Op}\sum_{j=0}^{i}C_{ij}\,x^j\,y^{i-j} \tag{4}$$

In this formula C_{ij} represent the coefficients of the interpolation polynomial; x and y represent the coordinates of the intersection point of the optical ray, considered by us, with the reference sphere of the investigated optical system, and Op represents the order of the polynomial.

The determination of the coefficients for these polynomials is made using the following method.

We assume a uniform distribution, with M points, in the entry pupil. For each point from this distribution, we calculate the N components ($x^j y^{i-j}$) of the interpolation polynomial, which are written as a_{ij}. Next we determine the values of the unknown terms C_{ij}, written with x_j, $j=1..N$, so that, the difference between the value of the wavefront aberration calculated with the interpolation polynomial and the value calculated with the exact formula, written with b_i $i=1..M$, will represent a minimum regardless of the point from the entry pupil for which the investigation is performed. For this is defined the difference:

$$d_{ij}=\sum_{j=1}^{N}a_{ij}\cdot x_j-b_i;\quad i=1..M \tag{5}$$

and the function

$$\Phi=\sum_{i=1}^{M}d_{ij}^2=\sum_{i=1}^{M}\left(\sum_{j=1}^{N}a_{ij}\,x_j-b_i\right)^2 \tag{6}$$

The values x_i that minimize this function are the solutions of the system obtained by canceling the partial derivatives of the function Φ from the formula 6. By applying the difference is obtained the expression 7.

$$\frac{\partial\Phi}{\partial x_k}=\sum_{i=1}^{M}\left[2\left(\sum_{j=1}^{N}a_{ij}\,x_j-b_i\right)a_{ik}\right]=2\sum_{i=1}^{M}\left[\sum_{j=1}^{N}a_{ij}\,a_{ik}\,x_j-b_i\,a_{ik}\right]=\quad 2\left[\sum_{j=1}^{N}\left(\sum_{i=1}^{M}a_{ij}\,a_{ik}\right)x_j-\sum_{i=1}^{M}b_i\,a_{ik}\right] \tag{7}$$

If we write:

$$A_{kj}=\sum_{i=1}^{M}a_{ij}a_{ik}$$

and $B_k=\sum_{i=1}^{M}b_i a_{ik}$

is obtained the formula 7 as

$$\frac{\partial\Phi}{\partial x_k}=2\left(\sum_{j=1}^{N}A_{kj}x_j-B_k\right)$$

We equate with zero each partial derivative and we obtain the linear system of N equations with N unknowns, from which are obtained the coefficients of the interpolation polynomial.

$$\begin{cases}\sum_{j=1}^{N}A_{kj}\,x_j-B_k=0\\ k=1..N\end{cases} \tag{8}$$

The illustration of the formula 3, apparently simple, will be made for the selected mathematical model (see figure 1), when assuming that the refractive index of the lens is constant within its body. The wavefront aberrations for the optical system of the eyeball in the meridian plane and in the sagittal plane are presented in figure 4.

For a greater accuracy of the interpolation polynomial were employed 120 optical rays with uniform distribution in the entry pupil, and the order of the interpolation polynomial is Op = 10.

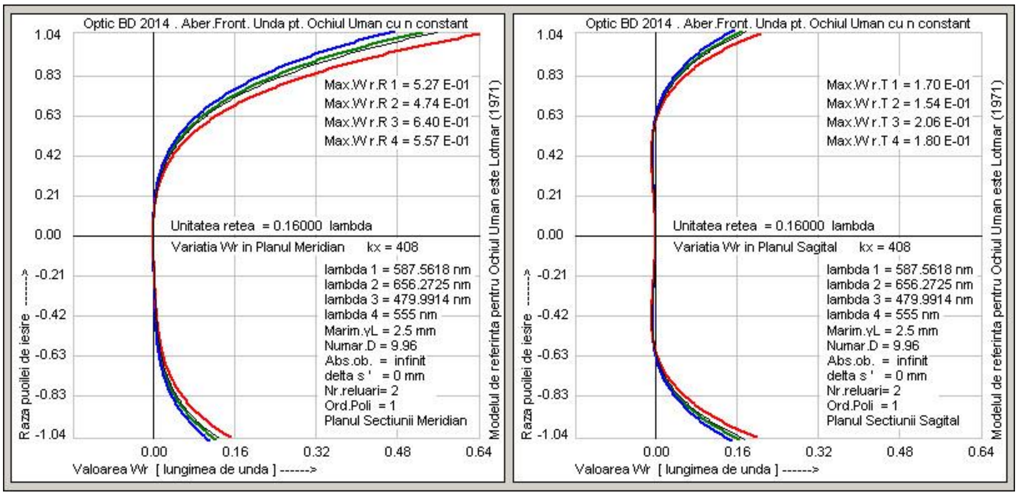

Fig. 4. The wavefront aberration for the optical system of the eyeball when the refractive index of the lens is constant throughout its body.

3.2. Modulation transfer function [6][7]

The wavefront aberration is also employed for the construction of the single link that can be presented in optics between the resolution and the contrast of an image of an optical system, meaning the optical transfer function (OTF). No single theory was established for the calculation of the optical transfer function. In most scientific specialty papers is employed the Hopkins method, which gives the formula for the module of the optical transfer function, the so called modulation transfer function (MTF), in the sagittal plane (formula 9):

$$D(s) = \frac{1}{A}\iint_S e^{ik\left[W\left(x+\frac{s}{2},y\right)-W\left(x-\frac{s}{2},y\right)\right]}\,dx\,dy \qquad (9)$$

and in the meridian plane (formula 10):

$$D(s) = \frac{1}{A}\iint_S e^{ik\left[W\left(x,y+\frac{s}{2}\right)-W\left(x,y-\frac{s}{2}\right)\right]}\,dx\,dy \qquad (10)$$

where: A = area of the exit pupil for the optical system.

s = reduced spatial frequency $\left(s = \dfrac{f \cdot \lambda}{apertura \quad numerica}\right)$;

f = real spatial frequency;

S = common area of the intersection between two exit pupils with axially symmetrical displacement with s/2;

k = wave number $\left(k = \dfrac{2\pi}{\lambda}\right)$;

W = wavefront aberration;

D(s) = modulation transfer function for the reduced spatial frequency.

The calculation method for these expressions is apparently simple, but it becomes complicated when it is closely examined.

Many of the problems with the calculation of the OTF regard the practical difficulties when evaluating the equation (9) and respectively (10), due to the complexity of obtaining the wavefront aberration.

The program used to calculate the optical transfer function has three phases:

1. Evaluation of the shape of the exit pupil, because in almost all cases it is affected by vignetting and directly influences the integration area. The integration area was approximated with an ellipsis.
2. The calculation of the polynomial of the wavefront aberration involved the use of a polynomial for the approximation of the wavefront aberration.
3. Calculation of the integral

3.3 Next is presented the calculation method for the modulation transfer function.

Many of the problems with the calculation of the OTF regard the practical difficulties when evaluating the equation (9) and respectively (10).

In this paper for the integration is employed the method of Gaussian quadrature, which is often used, due to the relatively small number of points required for the calculation.

Basically the method transforms the integral $\int_a^b f(x)\,dx$ to the integral $\frac{1}{2}(b-a)\int_{-1}^{+1} f(\xi)\,d\xi$, meaning it reduces the integration interval [a, b] to the interval [-1, 1] by employing the substitution formula $\xi = \dfrac{2x-(b+a)}{b-a}$.

The result can be written as

$$\frac{1}{2}(b-a)\int_{-1}^{+1} f(\xi)\,d\xi = \frac{1}{2}(b-a)\sum_{k=1}^{n} f(\xi_k)\,w_k \qquad (11)$$

where ξ_k and w_k are the coordinates respectively the weights of the integration method.

For example, for n = 6, the abscissas respectively the weights are presented in table 6.

A method of employing the Gaussian quadrature for the evaluation of the integrals (9) and (10) consists in the separation of the integral by x, from the integral by y. Thus if we write:

$$g(y) = \int e^{ik\left[w\left(x+\frac{s}{2},y\right)-w\left(x-\frac{s}{2},y\right)\right]}dx \qquad (12)$$

the integral (9) is calculated with the formula:

$$D(s) = \frac{1}{A}\int g(y)\,dy \qquad (13)$$

Of course, in the same manner is obtained:

$$h(x) = \int e^{ik\left[w\left(x,y+\frac{s}{2}\right)-w\left(x,y-\frac{s}{2}\right)\right]}dy \qquad (14)$$

and the integral (10) is calculated with the formula:

$$D(s) = \frac{1}{A}\int h(x)\,dx \qquad (15)$$

Table 6. Coordinates respectively weights of the integration method employing the Gaussian quadrature

	ξ	w
1	-0.9324	0.1713
2	-0.6612	0.3607
3	-0.2386	0.4679
4	0.2386	0.4679
5	0.6612	0.3607
6	0.9324	0.1713

The exemplification of the formulas 9 and 10, for the four radiations employed, is presented in Fig. 5.

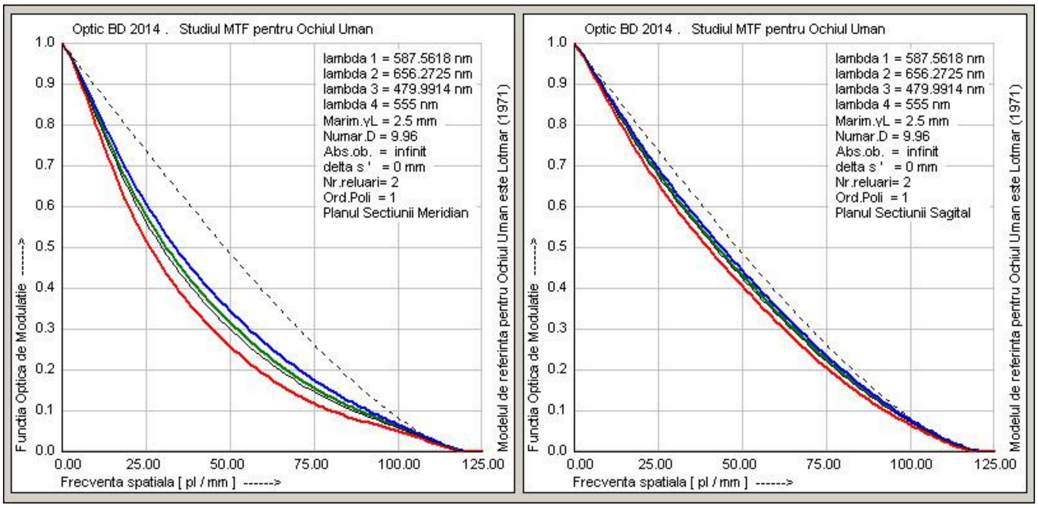

Fig. 5. Modulation transfer function (MTF) for the optical system of the eyeball when the refractive index of the lens is constant throughout its body.

4. Determination of the formula for the variation of the refractive index of the lens [2]

In the scientific specialty papers are presented proposals for methods for determination for the variation of the refractive index of the lens based on using aspherical diopters.

In this paper is proposed an approach based on the wavefront aberration for the optical system with spherical diopters. The determination of the formula for the variation of the refractive index of the lens, related to the main pupil radius and a given radius, is based on the optical diagram presented in figure 6.

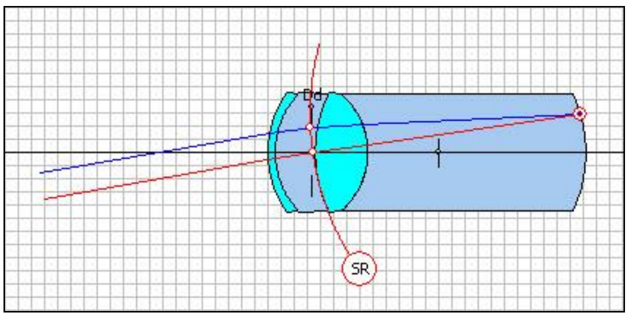

Fig. 6. Pathway of the main pupil radius and a given radius through the optical system of the eyeball

The definition formula 1 is written as:

$$\sum_{j=1}^{4}\left(n_j\overline{D}_j\right)+\left(n_5\overline{D}_{SR}\right)-\left[\sum_{j=1}^{4}\left(n_jD_j\right)_i+\left(n_5D_{SR}\right)_i\right]=kW_i \tag{16}$$

where: $n_j\overline{D}_j =$ Basic optical pathway along the main pupil radius,

$n_jD_j =$ Basic optical pathway along the radius investigated and which is part of the beam of optical rays,

$n_5\overline{D}_{SR} =$ Optical pathway of the main pupil radius from the last diopter to the reference sphere.

$\left(n_5D_{SR}\right)_i =$ Optical pathway of given radius i, from the last diopter to the reference sphere.

$k =$ The less than one factor imposed by the user, which indicates the number of times the wavefront aberration will be decreased depending on the radius involved.

$W_i =$ The wavefront aberration for the optical radius involved in the hypothesis of the constant refraction index for the lens throughout its body, having the value n_4= 1.42007. The expression 16 can also be written as:

$$\sum_{j=1}^{4}\left(n_j\overline{D}_j\right)+\left(n_5\overline{D}_{SR}\right)-\sum_{j=1}^{3}\left(n_jD_j\right)_i-\left(n_4D_4\right)_i-\left(n_5D_{SR}\right)_i=kW_i \tag{17}$$

From expression 17 can be determine the refraction index for the lens in the point where the involved radius pierces the plane tangent to the anterior surface of the lens.

$$n_{4\,i}=\frac{\sum_{j=1}^{4}\left(n_j\overline{D}_j\right)+\left(n_5\overline{D}_{SR}\right)-\sum_{j=1}^{3}\left(n_jD_j\right)_i-\left(n_5D_{SR}\right)_i-k\,W_i}{D_{4\,i}} \tag{18}$$

Expression 18 can be used to calculate the refraction index for all rays of the beam of optical rays that traverses the optical system of the eyeball i = 0. . N_{rays}, where N_{rays} represents the number of optical rays in the beam. The relation is applied to each ray belonging to the beam when the optical ray traverses the lens.

The confirmation of the proposal to change the refraction index is shown in figure 8. In the table from this figure, in the first column are shown the values of the wavefront aberrations calculated with a constant refraction index throughout the body of Wr, in the second column are shown the values of the wavefront aberrations calculated with variable values for the refraction index throughout the body of $W_{n\ variable}$, and in the third column are presented the refraction indexes calculated with expression 18.

On the right side of figure 7 is the table with the entry data involved in the calculation using expression 18. Next to this table are presented the minimum and maximum values for variation of the refraction index of the lens, the number of the point where these values are located, the radius of the entry pupil and the radius of the reference sphere employed for these calculations.

For the point with the maximum wavefront aberration is observed that W_{81}= 0.5269 wavelengths when the refraction index of the lens is constant and W_{81} = 0 wavelengths when the refraction index of the lens varies using the formula 18.

If we consider the Rayleigh criterion for assessing the resolution, which says that an optical system is considered aberration free if the aberration of the wavefront is smaller than 0.25 wavelengths, we can state that by applying the formula 18 the selected mathematical model of the eyeball, is an optical system free of aberrations.

For a complete example, next will be presented the wavelength aberrations for $y'_L = 2.5\,\text{mm}$. At this value on the eyeball is located the fovea, where the density of the photosensitive cells is maximum (from the 7 million cone cells most are concentrated on the surface of the fovea with an elliptical shape with the long axis of 2 mm and the short axis of 1 mm).

The curves are traced for the standard radiations n_C, n_D, n_F si n_g for an opening number $N_d = 9.96$, which leads to the diameter of the iris equal with D = 2.23 mm and the object abscissa as infinite. The value of the diameter of the pupil is calculated at the optimal and environmental lighting. The analysis was conducted in the meridian plane and the sagittal plane.

In fig. 8 are presented the curves of the wavefront aberration obtained with the index of the lens variable throughout its body, from the values given by formula 18.

In fig. 8 is represented graphically the wavefront aberration for the optical system of the eyeball when the refractive index of the lens is variable throughout its body.

It can be observed a null value for the wavefront aberration.

In fig. 9 are traced the graphs for the variation of the refractive index of the lens to obtain an optical system of the eyeball without aberrations, and in figure 10 are presented the MTF curves for this model.

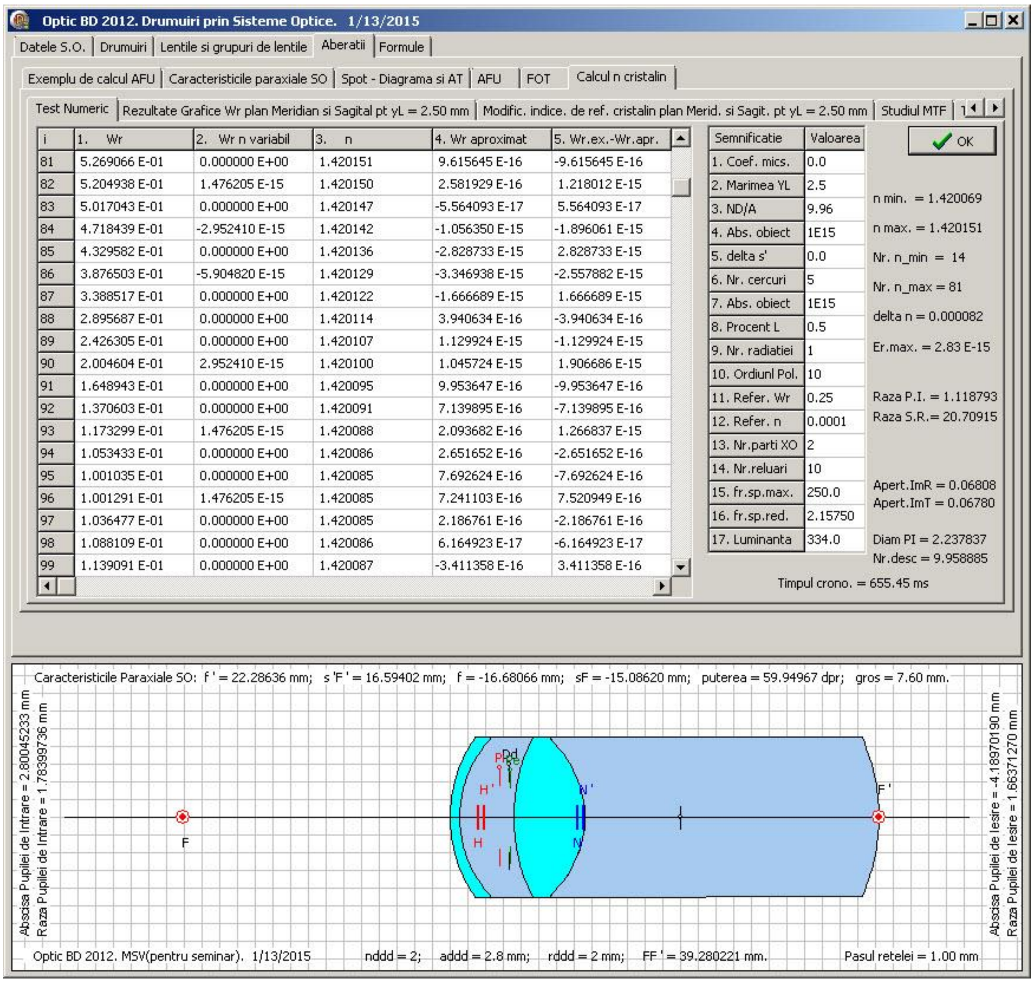

Fig. 7. Test for validation of formula 6 for variation of the refractive index of the lens having the refractive index of the lens variable throughout its body.

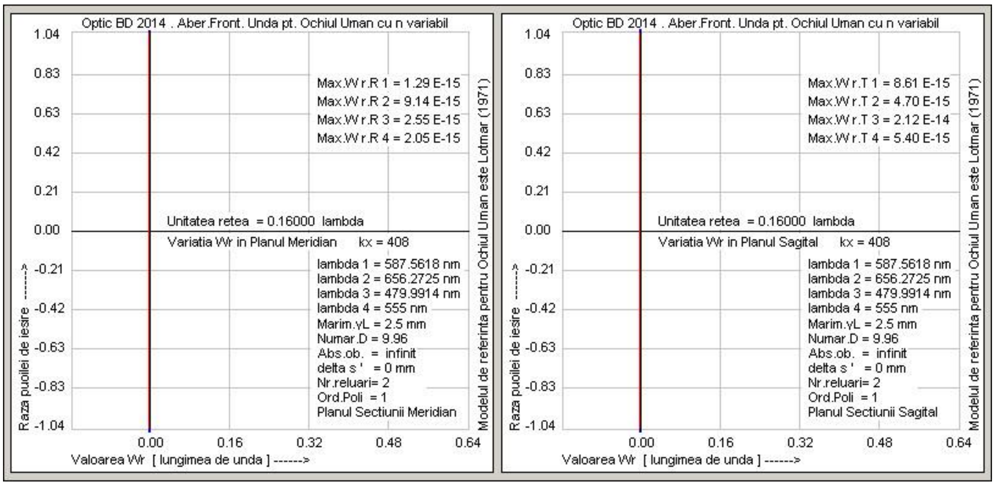

Fig. 8. The wavefront aberration for the optical system of the eyeball when the refractive index of the lens is variable throughout its body.

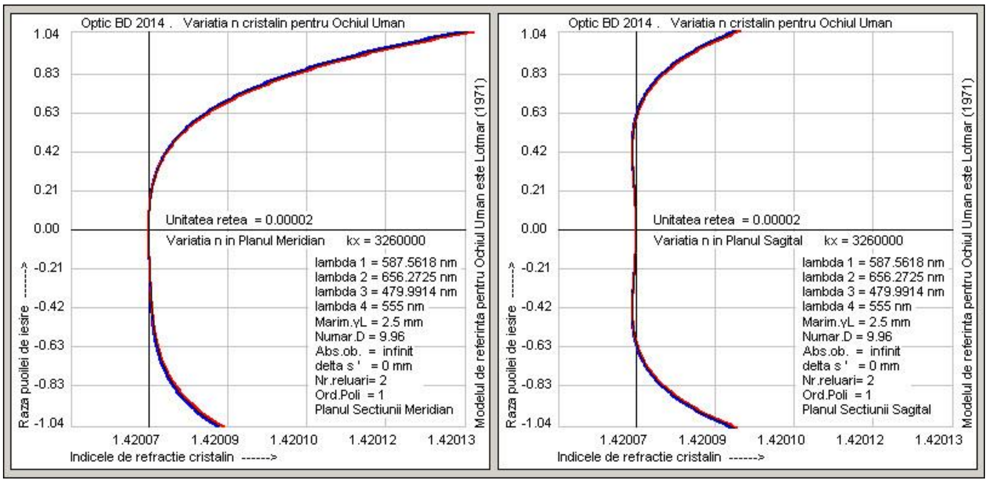

Fig. 9. Variation of the refractive index of the lens in the optical system of the eyeball

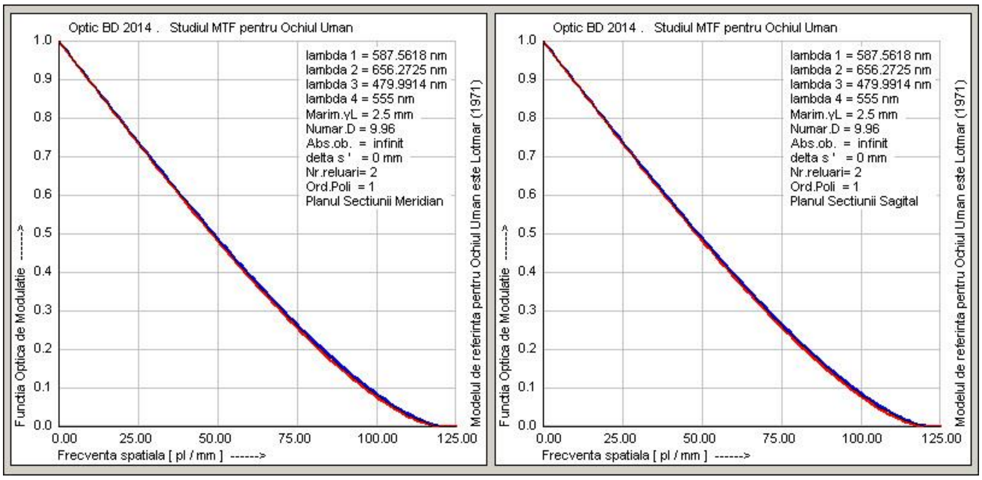

Fig. 10 Modulation transfer function (MTF) for the optical system of the eyeball when the refractive index of the lens is variable throughout its body.

5. Conclusions

The calculation of the complex aberrations for the optical system of the eyeball leads to high values, contradicting the real perception of the image as it is seen by the human eye. This is explained by the fact that the anatomy of the lens shows that the refraction index of the lens is not constant throughout its body and it is known that it varies from the optical axis to the edge and also along the optical axis. This type of modification is nature's solution for correcting the aberrations. Sadly the variation law for the refraction index throughout the lens body is not known. In order to calculate the extra-paraxial aberrations listed above, maintaining the concordance with the physical reality of the eye is imposed the determination of the variation law for the refraction index throughout the lens body (formula 18).

The analysis of the aberrations of the optical system provides the information regarding the radius of the eyeball. With this radius can be considered that the mathematical model of the emmetropic eye is obtained.

On this model can be changed the radius of the eyeball obtaining an ametropic eye, which can be corrected with an eyeglass lens or a contact lens.

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