Computing the metric and partition dimension of H-Naphtalenic and VC_5C_7 nanotubes

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It is well known that the relation between metric dimension and partition dimension of a non-trivial connected graph G denoted by $dim(G)$ and $pd(G)$, respectively is given by the following inequality: $pd(G) \le dim(G)+1$.However, the metric dimension of a connected graph G may be much larger than its partition dimension and this phenomena is called a discrepancy between metric dimension and partition dimension.In this paper, we study the metric dimension (location number) and partition dimension of 2-dimensional lattices of *H-Naphtalenic* and VC_5C_7 infinite nanotubes generated by tiling of the plane. We prove that the metric dimension of these two infinite nanotubes is not finite but their partition dimension is three, implying that these nanotubes are among the graphs having discrepancy between their metric dimension and partition dimension. It is natural to ask about characterization of the graphs having discrepancies between their metric dimension and partition dimension. Furthermore, it is also proved that there exist induced subgraphs of 2-dimensional lattices of these two nanotubes some of them have metric dimension depending upon n and others have constant metric dimension.

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1.Introduction and preliminary results

Navigation can be studied in a graph-structured framework in which the navigation agent (which we shall assume to be a point robot) moves from node to node of a “graph space". The robot can locate itself by the presence of distinctly labeled “landmarks" nodes in the graph space. For a robot navigating in Euclidean space, visual detection of a distinctive landmark provides the information about the direction to the landmark, and allows the robot to determine its position by triangulation. On a graph, however, there is neither the concept of direction nor that of visibility. Instead, we shall assume that a robot navigating on a graph can sense the distances to a set of landmarks.

Evidently, if the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph, what are the fewest number of landmarks needed, and where they should be located, so that the distances to the landmarks uniquely determine the robot’s position on the graph? This is actually a classical problem about metric spaces. A minimum set of landmarks which uniquely determines the robot’s position is called a “metric basis", and the minimum number of landmarks is called the “metric dimension" of a graph.

The *distance* $d(x, y)$ between two vertices $x, y \in V(G)$ in a connected graph G is the length of a shortest path between them and diameter of a connected graph G is $\max_{x,y\in V(G)} d(x, y)$. An ordered set of vertices $W \subseteq V(G)$ is called a *resolving set* or *locating set* for G if every vertex can be uniquely identified by its vector of distances to the vertices in W. The *metric dimension* or *location number* of G is the minimum cardinality of a resolving set of G, denoted by $dim(G)$ or $loc(G)$.

To prove that W is a resolving set it is sufficient to verify that $r(x|W) \ne r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\setminus W$.

The following lemma is very useful in finding $dim(G)$.

Lemma 1.*[26] Let W be a resolving set for a connected graph G and $x, y \in V(G)$. If $d(x,z) = d(y,z)$ for all vertices $z \in V(G)\setminus\{x, y\}$, then $\{x, y\}\cap W \ne \emptyset$.*

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n\ge 1}$ depending on n as follows: the order $|V(G_n)| = \varphi(n)$ and $\lim_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G_n) \le C$ for every $n \ge 1$, then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.

If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension. The families of graph having constant metric dimension were discussed previously in

[17, 18, 19].

Another kind of dimension called the partition dimension of a connected graph G denoted by $pd(G)$ was introduced in [6, 7] as a natural generalization of metric dimension as follows: The distance $d(v,S)$ between v and S is usually defined as $d(v,S)=\min\{d(v,x):x\in S\}$, where For $S\subset V(G)$ and $v\in V(G)$. If $\Pi=(S_1,S_2,....,S_k)$ is an ordered k-partition of $V(G)$, the representation of v with respect to Π is the k-tuple $r(v\mid\Pi)=(d(v,S_1),d(v,S_2),.....,d(v,S_k))$. If the k-tuples $r(v\mid\Pi)$ for $v\in V(G)$ are all distinct, then the partition Π is called a resolving partition and the minimum cardinality of a resolving partition of $V(G)$ is called the partition dimension of G. To determine whether a given partition Π of $V(G)$ is a resolving partition for $V(G)$, we need only to verify if the vertices of G belonging to the same class of Π have distinct representations with respect to Π. When $d(u,S_i)\neq d(v,S_i)$ we shall say that the class S_i distinguishes vertices u and v. Another useful property in determining $pd(G)$ is the following lemma [7].

Lemma 2. *Let Π be a resolving partition of $V(G)$ and $u,v\in V(G)$. If $d(u,w)=d(v,w)$ for all vertices $w\in V(G)\setminus\{u,v\}$, then u and v belong to different classes of Π.*

In [6] it was shown that for any nontrivial connected graph G we have $pd(G)\leq dim(G)+1$. However, the metric dimension may be much larger than the partition dimension, and this phenomena is known as discrepancy between metric dimension and the partition dimension and this phenomena is called a discrepancy between their metric dimension and partition dimension (see [26]).

These concepts have some applications in chemistry for representing chemical compounds (see [5]) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures and structure-activity maps for visualizing the graph variable arising in drug design (see [8], [20] and [21]).

Carbon *nanotubes* are basically sheets of graphite rolled up into a tube. It is constructed from the hexagonal two dimensional lattice of graphite mapped on a given one-dimensional cylinder of radius R. The nanotube H-Naphtalenic is constructed in a similar way by a sheet covered by squares, hexagons and octagons. Also the nanotube VC_5C_7 is constructed by a sheet covered by pentagons and heptagons (see Fig. 1).

The nanotubes have been extensively studied with respect to different graph-theoretic parameters in graph theory like chromatic polynomials, symmetry groups, bipartite edge frustration, Padmakar-Ivan, Omega and Sadhana Polynomial of $HAC_5C_6C_7$ nanotubes, szeged index, balaban index of an armchair polyhex and topological indices, to name a few (see [1], [2], [3], [9], [11], [13, 14, 15, 16] and [27]). The metric and partition dimension of infinite nanotubes HAC_5C_7, HC_5C_7 and $HAC_5C_6C_7$ has been computed recently in [22] where it is proved that these infnite nanotubes have discrepancy between their metric dimension and partition dimension.

In what follows we shall consider two infinite regular graphs generated by tiling of the plane by 2-dimensional lattices of H-Naphtalenic and VC_5C_7 infinite nanotubes. In this paper, we extend this study to 2-dimensional lattices of H-Naphtalenic and VC_5C_7 nanotubes and prove that these two nanotubes have also discrepancy between their metric dimension and partition dimension.

2. Main results

In this section, we compute the metric dimension and the partition dimension of infinite regular graphs generated by tiling of the plane by 2-dimensional lattices of H-Naphtalenic and VC_5C_7 infinite nanotubes. We show that these two 2-dimensional lattices of nanotubes have no finite metric bases but their partition dimension is finite. These results show that these two 2-dimensional lattices of nanotubes are among the graphs for which the strict inequality $pd(G)<dim(G)+1$ holds thus proving that these nanotubes have discrepancy between their metric dimension and partition dimension. It is also shown that there exist some induced subgraphs of 2-dimensional lattices of these nanostructures having metric dimension depending upon n as well as having constant metric dimension.

In the next lemma, we prove that the 2-dimensional lattices of infinite nanotubes H-Naphtalenic and VC_5C_7 have infinite metric dimension.

Lemma 3. *The 2-dimensional lattices of infinite nanotubes H-Naphtalenic and VC_5C_7 have no finite metric basis, i.e.,*

$$dim(H\text{ - }Naphtalenic)=dim(VC_5C_7)=\infty.$$

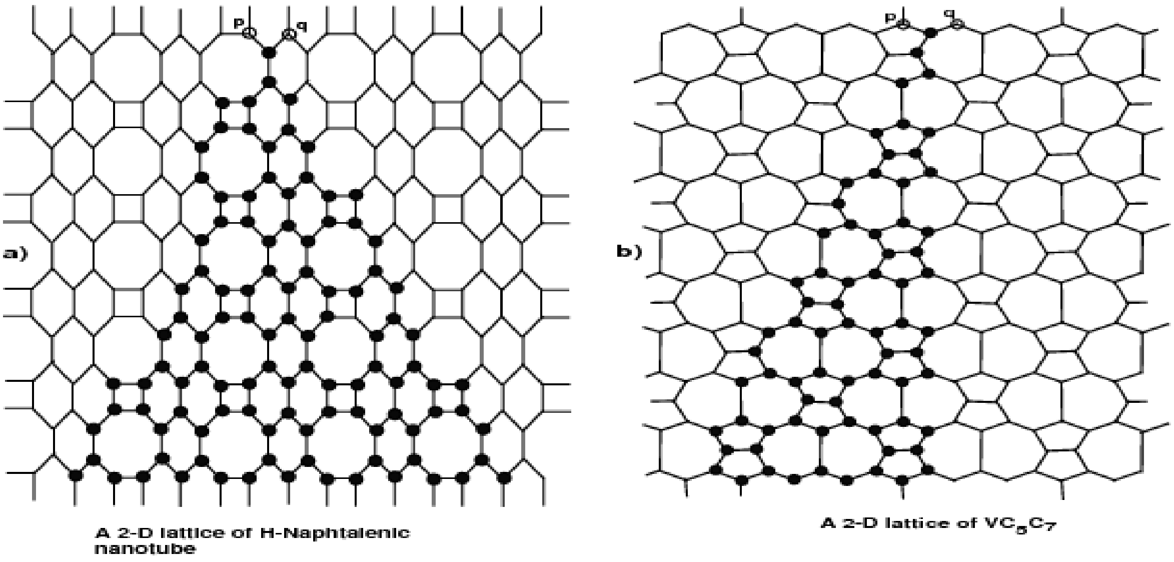

Fig. 1: Vertices having equal distances from p and q

Proof. Consider the graph of 2-dimensional lattice of H-Naphtalenic nanotube as shown in Fig. 1 a), where we labeled two vertices by p and q and some vertices r in this graphs such that $d(p,r) = d(q,r)$.

On contrary, suppose that this graph has finite metric basis L. We can find two vertices p, q and a subset M of this graph consisting of all vertices r such that $d(p,r) = d(q,r) \le m$ for every positive integer m such that $L \subset M$. This implies that $d(p,r) = d(q,r)$ for all $r \in L$, a contradiction to our assumption. The proof is similar for the 2-dimensional lattice of VC_5C_7 nanotube.

Now we compute the partition dimension of 2-dimensional lattices of H-Naphtalenic and VC_5C_7 infinite nanotubes in the following lemma.

Lemma 4. *We have pd(H-Naphtalenic)* $= pd(VC_5C_7) = 3.$

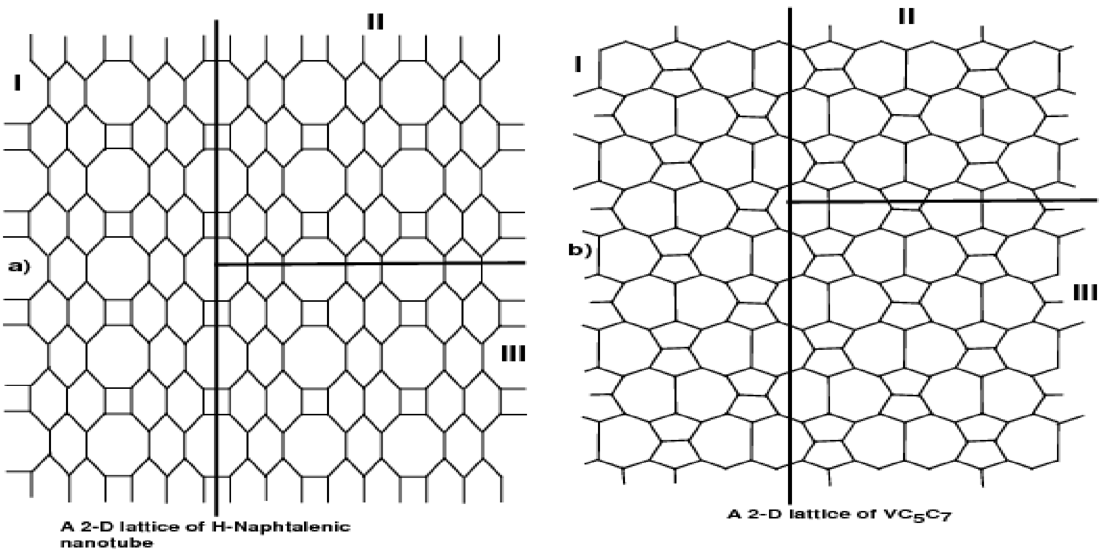

Fig. 2: A resolving 3-partition of 2-dimensional lattices of H-Naphtalenic and VC_5C_7 *infinite nanotubes*

Proof. It was proved by Chartrand et.al in [6] that $pd(G) = 2$ if and only if G is path and this property is also valid for infinite graphs. It follows that $pd($H-Naphtalenic$) \ge 3$ and $pd(VC_5C_7) \ge 3$.

Fig. 2 provides a resolving 3-partition of these 2-dimensional lattices of H-Naphtalenic and VC_5C_7 infinite nanotubes. This implies that pd (H-Naphtalenic) $= pd(VC_5C_7) = 3$, which completes the proof.

As an immediate consequence of Lemma 2 and Lemma 1, we deduce that the 2-dimensional lattices of H-Naphtalenic and VC_5C_7 infinite nanotubes have discrepancy their metric dimension and partition dimension.

A k*-polyomino* system is a finite 2-connected plane graph such that each interior face (also called cell) is surrounded by a regular 4k-cycle of length one. In other words, it is an edge-connected union of cells (see [10]).

Fig. 3 represents certain induced subgraphs of 2-dimensional lattices of H-Naphtalenic and VC_5C_7 nanotubes. The induced subgraph M_n is defined an edge-connected union of n pairs of induced 6-cycles and $n-1$ induced 4-cycles alternatively, R_n is defined as an edge-connected union of n pairs of induced 7-cycles and n-1 pairs of induced 4-cycles alternatively, BB_n is defined as an edge-connected union of n induced pairs of C_8 and C_6, and an edge alternatively, while the induced subgraph YY_n is defined as an edge connected union of n pairs of zig-zag C_6 and an edge alternatively. Note that all the defined induced subgraphs are of order l.

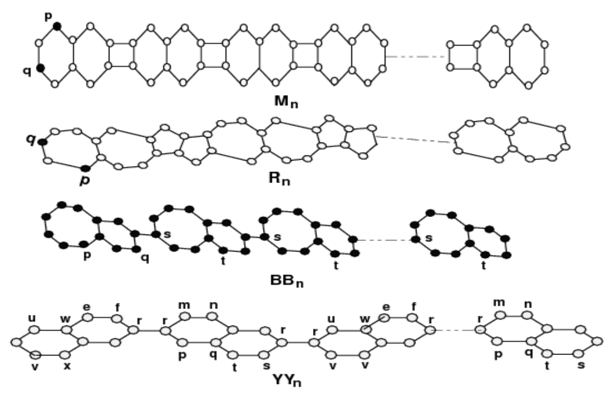

Fig. 3: Some induced subgraphs of nanotubes

In the next theorems, we determine the metric dimension of induced subgraphs defined above.

Theorem 1. a) *For every positive integer* $n \geq 2$, *we have*: $dim(M_n) = dim(R_n) = 2$;

b) *For every positive integer* $n \geq 2$ *we have*: $dim(BB_n) = dim(YY_n) = n$.

Proof. a) It was proved in [5] that $dim(G) = 1$ if and only if G is a path implying that $dim(M_n) \geq 2$. In Fig. 3, the vertices of M_n lying on the upper and lower half of the induced 6-cycles and 4-cycles are labeled by x_i and y_i respectively, where $1 \leq i \leq \frac{l}{2}$. To show that $dim(M_n) \leq 2$, we prove that $W_1 = \{x_2, y_1\}$ resolves $V(M_n)$. For this, we give representations of the vertices of $V(M_n)\setminus W_1$ with respect to W_1.

$r(x_1 \mid W_1) = (1,1)$, $r(y_2 \mid W_1) = (3,1)$.

Also

$$r(x_i \mid W_1) = (i-2, i), \quad \text{for} \quad 3 \leq i \leq \frac{l}{2},$$

and

$$r(y_i \mid W_1) = (i-1, i-1), \quad \text{for} \quad 3 \leq i \leq \frac{l}{2}.$$

This implies that $dim(M_n) = 2$. The proof of $\dim(R_n) = 2$ follows on the same lines and is therefore omitted.

b) It can be seen that there are n induced pairs containing octagons and hexagons in BB_n. Suppose that these octagonal-hexagonal pairs of BB_n have been numbered by $1,2,3,...,n$ from left to right.

Fig. 3 depicts that the vertices p and q of BB_n can only be distinguished by the vertices of the pair numbered by 1 in BB_n and the vertex of type s of the remaining pairs and have equal distance to all other vertices of BB_n if p or q do not belong to the metric basis of BB_n. This implies that at least one of them must be included in any metric basis of BB_n.

On the other hand, we can construct a metric basis of BB_n by taking only one vertex of type t from each pair numbered by $2,3,...,n$ of the induced subgraph BB_n and the result follows.

There are n pairs of zig-zag hexagons in the sequence of hexagons of the graph YY_n. The vertices u and v have equal distances to all vertices of YY_n different from w, x, e and f of first pairs of zig-zag hexagons, and the vertices w and x may be distinguished by the vertices u, v, e or f of YY_n, the situation is similar for all other pairs of hexagons. If u and v do not belong to basis of YY_n, it follows that at least one vertex from the set $\{u, v, w, x, e, f\}$ of YY_n must belongs to any metric basis of YY_n.

On the other hand by choosing exactly one vertex of degree two in each of pair of zig-zag hexagons of YY_n, these set of vertices form a metric basis for YY_n and the result follows.

3. Concluding remarks

In this paper, we have computed the metric dimension (location number) and partition dimension of 2-dimensional lattices of infinite version of H-Naphtalenic and VC_5C_7 nanotubes. We proved that the 2-dimensional lattices of these infinite nanotubes have discrepancy between their metric dimension and partition dimension. It is natural to ask about the characterization of the graphs having discrepancies between their metric dimension and the partition dimension.

It is also shown that there exist some inducedsubgraphs of these 2-dimensional lattices of H-Naphtalenic and VC_5C_7 nanotubes such that some of them have metric dimension depending upon n and others have constant metric dimension.

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