Cubic–quartic solitons in couplers with optical metamaterials having generalized Kudryashov's law of refractive index

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The paper studies soliton solutions in couplers with optical metamaterials that are governed by the lately proposed Kudryashov's generalized form of power–law of refractive index. The low–count chromatic dispersion is also replenished with cubic–quartic dispersive effects. The couplers are of both kinds, namely twin–core couplers as well as multiple–core type. The auxiliary equation approach recovers a full spectrum of solitons that are exhibited with their necessary parameter restrictions.

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1. Introduction

The application of Kudryashov's newly proposed generalized power–law of refractive index has attracted wide attention across the globe [1–20]. There are several papers that have been reported with this proposed form of self–phase modulation effect [8–15]. There are applications that are visible in optical fibers, PCF, Bragg gratings as well as other kinds of optoelectronic devices. Today's work is an application of Kudryashov's generalized law of SPM to yet another device, namely optical couplers. The usual chromatic dispersion is also replaced by a combination of third–order dispersion (3OD) and fourth–order dispersion (4OD) effects. This is collectively known as cubic–quartic (CQ) dispersive effect. This is an occasional replacement is carried out when CD is on the verge of depletion.

To study the model from an updated standpoint, the material is considered to be of optical metamaterials. Thus, the governing model is the study of solitons in optical couplers that comprises of optical metamaterials and maintains Kudryashov's lately proposed law of refractive index. The new auxiliary equation approach is the integration scheme that is adopted in this work to recover soliton solutions. The solitons are also classified and are exhibited in the paper along with their existence criteria. The couplers that are considered are of twin–core type as well as multiple–core types. But multiple—core couplers are of two types that are studied in this paper, namely when the coupling is only with nearest neighbors or with all neighbors. The comprehensive set of results are exhibited after introductory discussions for each type of coupler.

2. Twin–core couplers

The governing system of the CQ twin-core couplers in optical metamaterials having the generalized Kudryashov's law of arbitrary refractive index with perturbation terms is written, for the first time, as:

$$iq_t + ia_1 q_{xxx} + b_1 q_{xxxx} + \left[\frac{c_1}{|q|^{4n}} + \frac{d_1}{|q|^{3n}} + \frac{e_1}{|q|^{2n}} + \frac{f_1}{|q|^n} + g_1|q|^n + h_1|q|^{2n} + k_1|q|^{3n} + l_1|q|^{4n}\right]q$$
$$= \alpha_1(|q|^2 q)_{xx} + \beta_1|q|^2 q_{xx} + \gamma_1 q^2 q^*_{xx} + \chi_1 r + i[\lambda_1(|q|^2 q)_x + \mu_1(|q|^2)_x q + \theta_1|q|^2 q_x], \quad (1)$$

and

$$ir_t + ia_2 r_{xxx} + b_2 r_{xxxx} + \left[\frac{c_2}{|r|^{4n}} + \frac{d_2}{|r|^{3n}} + \frac{e_2}{|r|^{2n}} + \frac{f_2}{|r|^n} + g_2|r|^n + h_2|r|^{2n} + k_2|r|^{3n} + l_2|r|^{4n}\right]r$$
$$= \alpha_2(|r|^2 r)_{xx} + \beta_2|r|^2 r_{xx} + \gamma_2 r^2 r^*_{xx} + \chi_2 q + i[\lambda_2(|r|^2 r)_x + \mu_2(|r|^2)_x r + \theta_2|r|^2 r_x]. \quad (2)$$

In Eqs. (1) and (2), the complex-valued soliton profiles are $q(x,t)$ and $r(x,t)$, where x and t are the independent variables that are respectively spatial and temporal components. Next, a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j, k_j, l_j, α_j, β_j, γ_j, χ_j, λ_j, μ_j and θ_j, $(j = 1,2)$ are real constants, while $i = \sqrt{-1}$. The coefficients of third-order dispersion and fourth–order dispersion are a_j and b_j, respectively. The constants c_j, d_j, e_j, f_j, g_j, h_j, k_j and l_j are from SPM effect, while n is the power nonlinearity parameter in SPM. Next, the coefficients of metamaterials are α_j, β_j and γ_j, while the coefficients of the coupling are χ_j. Finally, λ_j are the self-steepening terms, while μ_j and θ_j are the coefficients of nonlinear dispersion.

To retrieve solutions of Eqs. (1) and (2), the following hypothesis will be assumed:

$$q(x,t) = \Phi_1(\xi)e^{i(-\kappa x + \omega t + \varepsilon_0)},$$
$$r(x,t) = \Phi_2(\xi)e^{i(-\kappa x + \omega t + \varepsilon_0)}, \quad (3)$$

and

$$\xi = x - vt, \quad (4)$$

where $\kappa, \omega, \varepsilon_0$ and v are nonzero real constants. Here, κ is the soliton frequency, ω is the wave number, ε_0 is the phase constant and v is the soliton velocity. Next, the real functions $\Phi_j (j = 1,2)$ represent the amplitudes of wave transformation.

Inserting (3) along with (4) into Eqs. (1) and (2), then the real parts are given by:

$$b_1\Phi_1^{(4)} + 3\kappa(a_1 - 2b_1\kappa)\Phi_1'' - (\beta_1 + \gamma_1 + 3\alpha_1)\Phi_1^2\Phi_1'' - 6\alpha_1\Phi_1\Phi_1'^2 - (\omega + a_1\kappa^3 - b_1\kappa^4)\Phi_1 - \chi_1\Phi_2 + [\kappa^2(\beta_1 + \alpha_1 + \gamma_1) + \kappa(\lambda_1 + \theta_1)]\Phi_1^3 + c_1\Phi_1^{1-4n} + d_1\Phi_1^{1-3n} + e_1\Phi_1^{1-2n} + f_1\Phi_1^{1-n} + g_1\Phi_1^{1+n} + h_1\Phi_1^{1+2n} + k_1\Phi_1^{1+3n} + l_1\Phi_1^{1+4n} = 0, \quad (5)$$

and

$$b_2\Phi_2^{(4)} + 3\kappa(a_2 - 2b_2\kappa)\Phi_2'' - (\beta_2 + \gamma_2 + 3\alpha_2)\Phi_2^2\Phi_2'' - 6\alpha_2\Phi_2\Phi_2'^2 - (\omega + a_2\kappa^3 - b_2\kappa^4)\Phi_2 - \chi_2\Phi_1 + [\kappa^2(\beta_2 + \alpha_2 + \gamma_2) + \kappa(\lambda_2 + \theta_2)]\Phi_2^3 + c_2\Phi_2^{1-4n} + d_2\Phi_2^{1-3n} + e_2\Phi_2^{1-2n} + f_2\Phi_2^{1-n} + g_2\Phi_2^{1+n} + h_2\Phi_2^{1+2n} + k_2\Phi_2^{1+3n} + l_2\Phi_2^{1+4n} = 0, \quad (6)$$

while, the imaginary parts are given by:

$$(a_1 - 4b_1\kappa)\Phi_1''' + [2\kappa(\beta_1 - \gamma_1 + 3\alpha_1) + (3\lambda_1 + 2\mu_1 + \theta_1)]\Phi_1^2\Phi_1' - (v + 3a_1\kappa^2 - 4b_1\kappa^3)\Phi_1' = 0, \quad (7)$$

and

$$(a_2 - 4b_2\kappa)\Phi_2''' + [2\kappa(\beta_2 - \gamma_2 + 3\alpha_2) + (3\lambda_2 + 2\mu_2 + \theta_2)]\Phi_2^2\Phi_2' - (v + 3a_2\kappa^2 - 4b_2\kappa^3)\Phi_2' = 0. \quad (8)$$

Using the linearly independent principle on Eqs. (7) and (8), one gets the soliton frequency as:

$$\kappa = \frac{a_j}{4b_j}, \quad (9)$$

the soliton velocity:

$$v = (4b_j\kappa - 3a_j)\kappa^2, \quad (10)$$

and the constraint conditions

$$2\kappa(\beta_j - \gamma_j + 3\alpha_j) + (3\lambda_j + 2\mu_j + \theta_j) = 0. \quad (11)$$

Set

$$\Phi_2 = A\Phi_1, \quad (12)$$

such that $A \neq 0$ or 1. Consequently, Eqs. (5) and (6) take the forms:

$$b_1\Phi_1^{(4)} + 3\kappa(a_1 - 2b_1\kappa)\Phi_1'' - (\beta_1 + \gamma_1$$

$$+3\alpha_1)\Phi_1^2\Phi_1'' - 6\alpha_1\Phi_1\Phi_1'^2 - (\omega + a_1\kappa^3 - b_1\kappa^4 + A\chi_1)\Phi_1 + [\kappa^2(\beta_1 + \alpha_1 + \gamma_1) + \kappa(\lambda_1 + \theta_1)]\Phi_1^3 + c_1\Phi_1^{1-4n} + d_1\Phi_1^{1-3n} + e_1\Phi_1^{1-2n} + f_1\Phi_1^{1-n} + g_1\Phi_1^{1+n} + h_1\Phi_1^{1+2n} + k_1\Phi_1^{1+3n} + l_1\Phi_1^{1+4n} = 0, \tag{13}$$

and

$$b_2A\Phi_1^{(4)} + 3\kappa A(a_2 - 2b_2\kappa)\Phi_1'' - (\beta_2 + \gamma_2 + 3\alpha_2)A^3\Phi_1^2\Phi_1'' - 6\alpha_2A^3\Phi_1\Phi_1'^2 - [(\omega + a_2\kappa^3 - b_2\kappa^4)A + \chi_2]\Phi_1 + A^3[\kappa^2(\beta_2 + \alpha_2 + \gamma_2) + \kappa(\lambda_2 + \theta_2)]\Phi_1^3 + c_2A^{1-4n}\Phi_1^{1-4n} + d_2A^{1-3n}\Phi_1^{1-3n} + e_2A^{1-2n}\Phi_1^{1-2n} + f_2A^{1-n}\Phi_1^{1-n} + g_2A^{1+n}\Phi_1^{1+n} + h_2A^{1+2n}\Phi_1^{1+2n} + k_2A^{1+3n}\Phi_1^{1+3n} + l_2A^{1+4n}\Phi_1^{1+4n} = 0. \tag{14}$$

Eqs. (13) and (14) have the same form under the following conditions:

$$b_1 = Ab_2, \alpha_1 = \alpha_2A^3, c_1 = c_2A^{1-4n},$$
$$a_1 - 2b_1\kappa = A(a_2 - 2b_2\kappa), d_1 = d_2A^{1-3n},$$
$$\beta_1 + \gamma_1 + 3\alpha_1 = A^3(\beta_2 + \gamma_2 + 3\alpha_2),$$
$$e_1 = e_2A^{1-2n}, f_1 = f_2A^{1-n}, \qquad g_1 = g_2A^{1+n},$$
$$h_1 = h_2A^{1+2n}, k_1 = k_2A^{1+3n}, l_1 = l_2A^{1+4n}$$
$$\omega + a_1\kappa^3 - b_1\kappa^4 + A\chi_1 = (\omega + a_2\kappa^3 - b_2\kappa^4)A + \chi_2,$$
$$\kappa^2(\beta_1 + \alpha_1 + \gamma_1) + \kappa(\lambda_1 + \theta_1) = A^3[\kappa^2(\beta_2 + \alpha_2 + \gamma_2) + \kappa(\lambda_2 + \theta_2)]. \tag{15}$$

From (15), we have:

$$a_2 = \frac{a_1}{A}, b_2 = \frac{b_1}{A}, c_2 = \frac{c_1}{A^{1-4n}},$$
$$d_2 = \frac{d_1}{A^{1-3n}}, e_2 = \frac{e_1}{A^{1-2n}}, f_2 = \frac{f_1}{A^{1-n}},$$
$$g_2 = \frac{g_1}{A^{1+n}}, h_2 = \frac{h_1}{A^{1+2n}}, k_2 = \frac{k_1}{A^{1+3n}},$$
$$l_2 = \frac{l_1}{A^{1+4n}}, \alpha_2 = \frac{\alpha_1}{A^3}, \chi_2 = (\chi_1 - \omega)A + \omega,$$
$$\beta_2 = \frac{\beta_1 + \gamma_1 - \gamma_2A^3}{A^3}, \lambda_2 = \frac{\lambda_1 + \theta_1 - \theta_2A^3}{A^3}. \tag{16}$$

Balancing $\Phi_1^{(4)}$ and Φ_1^{4n+1} in Eq. (13), one gets the balance number $N = \frac{1}{n}$. Consequently, we take the following transformation:

$$\Phi_1(\xi) = U^{\frac{1}{n}}(\xi), \tag{17}$$

provided $U(\xi) > 0$. Substituting (17) into Eq. (13), results the following equation:

$$n^3b_1U^3U^{(4)} - 4n^2(n-1)b_1U^2U'U''' - 3n^2 \times (n-1)b_1U^2U''^2 + 6n(2n-1)(n-1)b_1 \times UU'^2U'' - (n-1)(2n-1)(3n-1)b_1U'^4 + 3n^2\kappa(a_1 - 2b_1\kappa)[nU^3U'' - (n-1)U^2U'^2] + n^4c_1 + n^4d_1U + n^4e_1U^2 + n^4f_1U^3 - n^4(\omega + a_1\kappa^3 - b_1\kappa^4 + A\chi_1)U^4 + n^4g_1U^5 + n^4h_1U^6 + n^4k_1U^7 + n^4l_1U^8 + n^4[\kappa^2(\beta_1 + \alpha_1 + \gamma_1) + \kappa(\lambda_1 + \theta_1)]U^{\frac{2+4n}{n}} - n^3(\beta_1 + \gamma_1 + 3\alpha_1) \times U^{\frac{2+3n}{n}}U'' + n^2[(n-1)(\beta_1 + \gamma_1) - 3(n-3)\alpha_1]U^{\frac{2+2n}{n}}U'^2 = 0. \tag{18}$$

For integrability, one must select

$$(n-1)(\beta_1 + \gamma_1) - 3(n-3)\alpha_1 = 0,$$
$$\kappa(\beta_1 + \alpha_1 + \gamma_1) + \lambda_1 + \theta_1 = 0,$$
$$\beta_1 + \gamma_1 + 3\alpha_1 = 0. \tag{19}$$

Consequently, Eq. (18) changes to:

$$n^3b_1U^3U^{(4)} - 4n^2(n-1)b_1U^2U'U''' - 3n^2 \times (n-1)b_1U^2U''^2 + 6n(2n-1)(n-1)b_1 \times UU'^2U'' - (n-1)(2n-1)(3n-1)b_1U'^4 + 3n^2\kappa(a_1 - 2b_1\kappa)[nU^3U'' - (n-1)U^2U'^2] + n^4c_1 + n^4d_1U + n^4e_1U^2 + n^4f_1U^3 - n^4(\omega + a_1\kappa^3 - b_1\kappa^4 + A\chi_1)U^4 + n^4g_1U^5 + n^4h_1U^6 + n^4k_1U^7 + n^4l_1U^8 = 0. \tag{20}$$

Now, we will solve Eq. (20) by using the following method:

2.1. New auxiliary equation approach

According to this method [20], we assume that Eq. (20) has the formal solutions:

$$U(\xi) = \sum_{L=0}^{N} \Omega_L F^L(\xi), \tag{21}$$

where $F(\xi)$ satisfies the first order auxiliary equation:

$$F'^2(\xi) = \sum_{J=0}^{8} S_J F^J(\xi). \tag{22}$$

Here $\Omega_L (L = 0,1,\dots,N)$ and $S_J (J = 0,1,\dots,8)$ are constants to be determined provided $\Omega_N \neq 0$ and $S_8 \neq 0$, where N is a positive integer. We determine the balance number N of (21) by using the homogeneous balance method as follows:

If $D(U) = N, D(U') = N + 3, D(U'') = N + 6$ and hence

$$D[U^s U^{(g)}] = N(s + 1) + 3g. \tag{23}$$

It is well known [19] that Eq. (22) has the following types of solutions:

Type–1: If $S_0 = S_1 = S_3 = S_4 = S_6 = S_7 = 0$, $S_5^2 - 4S_2S_8 > 0, S_2 > 0$, then Eq. (22) admits the bright soliton solutions:

$$F(\xi) = \left(\frac{2\epsilon S_2}{\sqrt{S_5^2-4S_2S_8}\cosh(3\sqrt{S_2}\xi+\xi_{01})-\epsilon S_5}\right)^{\frac{1}{3}}, \tag{24}$$

where $\epsilon = \pm 1$ and ξ_{01} is arbitrary constant.

Type–2: If $S_0 = S_1 = S_3 = S_4 = S_6 = S_7 = 0$, $S_5^2 - 4S_2S_8 < 0$, $S_2 > 0$, then Eq. (22) admits the singular soliton solutions:

$$F(\xi) = \left(\frac{2\epsilon S_2}{\sqrt{-(S_5^2-4S_2S_8)}\sinh(3\sqrt{S_2}\xi+\xi_{02})-\epsilon S_5}\right)^{\frac{1}{3}}, \tag{25}$$

where $\epsilon = \pm 1$ and ξ_{02} is arbitrary constant.

Type–3: If $S_0 = S_1 = S_3 = S_4 = S_6 = S_7 = 0$, $S_5^2 - 4S_2S_8 = 0$, $S_2 > 0$, then Eq. (23) admits the dark and singular soliton solutions as:

$$F(\xi) = \left\{-\frac{S_2}{S_5}\left[1 \pm \tanh\left(\frac{3}{2}\sqrt{S_2}\xi + \xi_{03}\right)\right]\right\}^{\frac{1}{3}}, \tag{26}$$

and

$$F(\xi) = \left\{-\frac{S_2}{S_5}\left[1 \pm \coth\left(\frac{3}{2}\sqrt{S_2}\xi + \xi_{04}\right)\right]\right\}^{\frac{1}{3}}, \tag{27}$$

respectively, where ξ_{03} and ξ_{04} are arbitrary constants.

Balancing $U^3U^{(4)}$ and U^8 in Eq. (20) by using (23), one gets:

$$4N + 12 = 8N \Rightarrow N = 3. \tag{28}$$

It follows that Eq. (20) has the following solution form:

$$U(\xi) = \Omega_0 + \Omega_1 F(\xi) + \Omega_2 F^2(\xi) + \Omega_3 F^3(\xi), \tag{29}$$

where $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 are constants, provided $\Omega_3 \neq 0$.

Family–1: Substituting (29) and (22) into Eq. (20), one gets algebraic equations. As a result, if we set $S_0 = S_1 = S_3 = S_4 = S_6 = S_7 = 0$, in the obtained algebraic equations and solving it by using the Maple, one gets the following results:

$$\Omega_0 = \frac{1}{2}\left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_1 - 2b_1\kappa)^2}{4(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{\frac{1}{4}},$$

$$\Omega_3 = \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_1 - 2b_1\kappa)^2}{4(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{\frac{1}{4}},$$

$$\Omega_1 = \Omega_2 = 0, S_5 = 0,$$

$$S_2 = \frac{n^2\kappa(a_1 - 2b_1\kappa)}{12(2n^2 - 2n + 1)b_1},$$

$$S_8 = -\frac{n^2\kappa(a_1-2b_1\kappa)}{6(2n^2-2n+1)b_1}, \tag{30}$$

and

$$\omega = -\frac{9\kappa^2(37n^2 - 32n + 9)(a_1 - 2b_1\kappa)^2 + 32(2n^2 - 2n + 1)^2(a_1\kappa^3 - b_1\kappa^4 + A\chi_1)b_1}{32(2n^2 - 2n + 1)^2 b_1},$$

$$c_1 = -\frac{81\kappa^4(36n^6 - 49n^4 + 14n^2 - 1)(a_1 - 2b_1\kappa)^4}{4096(2n^2 - 2n + 1)^4 b_1^2 l_1},$$

$$d_1 = 0, e_1 = 0,$$

$$f_1 = \frac{9\kappa^2(3n^3 - 7n^2 + 3n - 2)(a_1 - 2b_1\kappa)^2}{32(2n^2 - 2n + 1)^2 b_1} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_1 - 2b_1\kappa)^2}{4(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{\frac{1}{4}},$$

$$g_1 = -\frac{27n\kappa^2(n^2 - 4)(a_1 - 2b_1\kappa)^2}{4(2n^2 - 2n + 1)^2 b_1} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_1 - 2b_1\kappa)^2}{4(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{-\frac{1}{4}},$$

$$h_1 = -\frac{9\kappa^2(3n^3 + 19n^2 + 20n + 4)(a_1 - 2b_1\kappa)^2}{4(2n^2 - 2n + 1)^2 b_1} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_1 - 2b_1\kappa)^2}{4(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{-\frac{1}{2}},$$

$$k_1 = \frac{9\kappa^2(6n^3+13n^2+9n+2)(a_1-2b_1\kappa)^2}{2(2n^2-2n+1)^2 b_1} \times \left[-\frac{9\kappa^2(6n^3+11n^2+6n+1)(a_1-2b_1\kappa)^2}{4(2n^2-2n+1)^2 b_1 l_1}\right]^{-\frac{3}{4}}, \quad (31)$$

provided $b_1 l_1 < 0$. By using the solutions (24) and (25) one gets the following cases of solutions for Eqs. (1) and (2):

Case–1: If $S_2 = \frac{n^2\kappa(a_1-2b_1\kappa)}{12(2n^2-2n+1)b_1} > 0$, then substituting (30) along with (24) into Eq. (29), one gets the bright soliton solutions as:

$$q(x,t) = \left[-\frac{9\kappa^2\left(6n^3+11n^2+6n+1\right)(a_1-2b_1\kappa)^2}{4(2n^2-2n+1)^2 b_1 l_1}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2} \pm \frac{\sqrt{2}}{2} \times \operatorname{sech}\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_1-2b_1\kappa)}{(2n^2-2n+1)b_1}} \times (x-vt)+\xi_{01}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x+\omega t+\varepsilon_0)}, \quad (32)$$

and

$$r(x,t) = A\left[-\frac{9\kappa^2\left(6n^3+11n^2+6n+1\right)(a_1-2b_1\kappa)^2}{4(2n^2-2n+1)^2 b_1 l_1}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2} \pm \frac{\sqrt{2}}{2} \times \operatorname{sech}\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_1-2b_1\kappa)}{(2n^2-2n+1)b_1}} \times (x-vt)+\xi_{01}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x+\omega t+\varepsilon_0)}. \quad (33)$$

Case–2: If $S_2 = \frac{n^2\kappa(a_1-2b_1\kappa)}{12(2n^2-2n+1)b_1} < 0$, then substituting (30) along with (25) into Eq. (29), one gets the periodic solutions as:

$$q(x,t) = \left[-\frac{9\kappa^2\left(6n^3+11n^2+6n+1\right)(a_1-2b_1\kappa)^2}{4(2n^2-2n+1)^2 b_1 l_1}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2} \pm \frac{\sqrt{2}}{2} \times \csc\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_1-2b_1\kappa)}{(2n^2-2n+1)b_1}} \times (x-vt)+\xi_{02}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x+\omega t+\theta_0)}, \quad (34)$$

and

$$r(x,t) = A\left[-\frac{9\kappa^2\left(6n^3+11n^2+6n+1\right)(a_1-2b_1\kappa)^2}{4(2n^2-2n+1)^2 b_1 l_1}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2} \pm \frac{\sqrt{2}}{2} \times \csc\left[\frac{n}{2}\sqrt{-\frac{3\kappa(a_1-2b_1\kappa)}{(2n^2-2n+1)b_1}} \times (x-vt)+\xi_{02}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x+\omega t+\varepsilon_0)}. \quad (35)$$

The solutions (32)-(35) are satisfied under the constraint conditions (31).

Family–2: Substituting (29) and (22) into Eq. (20), one gets algebraic equations. As a result, if we set $S_0 = S_1 = S_3 = S_4 = S_6 = S_7 = 0$ and $S_8 = \frac{S_5^2}{4S_2}$, in the obtained algebraic equations and solving it by using the Maple, one gets the following results:

$$\Omega_0 = \frac{1}{2}\left[-\frac{9\kappa^2(6n^3+11n^2+6n+1)(a_1-2b_1\kappa)^2}{(2n^2-2n+1)^2 b_1 l_1}\right]^{\frac{1}{4}},$$

$$\Omega_3 = \left[-\frac{9\kappa^2(6n^3+11n^2+6n+1)(a_1-2b_1\kappa)^2}{(2n^2-2n+1)^2 b_1 l_1}\right]^{\frac{1}{4}},$$

$$\Omega_1 = \Omega_2 = 0,$$

$$S_2 = \frac{n^2\kappa(a_1-2b_1\kappa)}{3(2n^2-2n+1)b_1},$$

$$S_5 = \frac{2n^2\kappa(a_1-2b_1\kappa)}{3(2n^2-2n+1)b_1}, \quad (36)$$

and

$$\omega = -\frac{9\kappa^2(3n^2-8n+1)(a_1-2b_1\kappa)^2 + 8b_1(2n^2-2n+1)^2(a_1\kappa^3-b_1\kappa^4+A\chi_1)}{8b_1(2n^2-2n+1)^2},$$

$$c_1 = -\frac{81\kappa^4(36n^6 - 49n^4 + 14n^2 - 1)(a_1 - 2b_1\kappa)^4}{256(2n^2 - 2n + 1)^4 b_1^2 l_1},$$
$$d_1 = 0, e_1 = 0, f_1 = 0, g_1 = 0, k_1 = 0,$$

$$h_1 = -\frac{12n\kappa(n+1)(a_1 - 2b_1\kappa)}{(2n^2 - 2n + 1)}\sqrt{-\frac{l_1}{(6n^3 + 11n^2 + 6n + 1)b_1}}, \quad (37)$$

provided $b_1 l_1 < 0$. By using the solutions (26) and (27) one gets the following cases of solutions for Eqs. (1) and (2):

Case–1: Substituting (36) along with (26) into Eq. (29), then one gets the dark soliton solutions as:

$$q(x,t) = \left[-\frac{9\kappa^2\begin{pmatrix}6n^3 + 11n^2 \\ +6n + 1\end{pmatrix}(a_1 - 2b_1\kappa)^2}{(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{\frac{1}{4n}}$$
$$\times \left\{\frac{1}{2}\tanh\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_1 - 2b_1\kappa)}{(2n^2 - 2n + 1)b_1}} \times (x - vt) + +\xi_{03}\right]\right\}^{\frac{1}{n}}$$
$$\times e^{i(-\kappa x + \omega t + \varepsilon_0)}, \quad (38)$$

and

$$r(x,t) = A\left[-\frac{9\kappa^2\begin{pmatrix}6n^3 + 11n^2 \\ +6n + 1\end{pmatrix}(a_1 - 2b_1\kappa)^2}{(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{\frac{1}{4n}}$$
$$\times \left\{\frac{1}{2}\tanh\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_1 - 2b_1\kappa)}{(2n^2 - 2n + 1)b_1}} \times (x - vt) + +\xi_{03}\right]\right\}^{\frac{1}{n}}$$
$$\times e^{i(-\kappa x + \omega t + \varepsilon_0)}, \quad (39)$$

provided $(2n^2 - 2n + 1)(a_1 - 2b_1\kappa)\kappa b_1 > 0$.

Case–2: Substituting (36) along with (27) into Eq. (29), then one gets the singular soliton solutions as:

$$q(x,t) = \left[-\frac{9\kappa^2\begin{pmatrix}6n^3 + 11n^2 \\ +6n + 1\end{pmatrix}(a_1 - 2b_1\kappa)^2}{(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{\frac{1}{4n}}$$
$$\times \left\{\frac{1}{2}\coth\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_1 - 2b_1\kappa)}{(2n^2 - 2n + 1)b_1}} \times (x - vt) + +\xi_{04}\right]\right\}^{\frac{1}{n}}$$
$$\times e^{i(-\kappa x + \omega t + \varepsilon_0)}, \quad (40)$$

and

$$r(x,t) = A\left[-\frac{9\kappa^2\begin{pmatrix}6n^3 + 11n^2 \\ +6n + 1\end{pmatrix}(a_1 - 2b_1\kappa)^2}{(2n^2 - 2n + 1)^2 b_1 l_1}\right]^{\frac{1}{4n}}$$
$$\times \left\{\frac{1}{2}\coth\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_1 - 2b_1\kappa)}{(2n^2 - 2n + 1)b_1}} \times (x - vt) + +\xi_{04}\right]\right\}^{\frac{1}{n}}$$
$$\times e^{i(-\kappa x + \omega t + \varepsilon_0)}, \quad (41)$$

provided $(2n^2 - 2n + 1)(a_1 - 2b_1\kappa)\kappa b_1 > 0$.

The solutions (38)-(41) are satisfied under the constraint conditions (37).

3. Multiple–core couplers (Coupling with nearest neighbors)

The CQ system for multiple–core couplers in optical metamaterials having the generaized Kudryashov's law of arbitrary refractive index with perturbation terms is written as:

$$i\left(q^{(s)}\right)_t + ia_s\left(q^{(s)}\right)_{xxx} + b_s\left(q^{(s)}\right)_{xxxx}$$
$$+\left[\frac{c_s}{|q^{(s)}|^{4n}} + \frac{d_s}{|q^{(s)}|^{3n}} + \frac{e_s}{|q^{(s)}|^{2n}}\right.$$
$$+\frac{f_s}{|q^{(s)}|^{n}} + g_s\left|q^{(s)}\right|^n + h_s\left|q^{(s)}\right|^{2n} + k_s\left|q^{(s)}\right|^{3n}$$
$$\left.+ l_s\left|q^{(s)}\right|^{4n}\right]q^{(s)} = \alpha_s\left(\left|q^{(s)}\right|^2 q^{(s)}\right)_{xx}$$
$$+\beta_s\left|q^{(s)}\right|^2\left(q^{(s)}\right)_{xx} + \gamma_s\left(q^{(s)}\right)^2\left(q^{(s)^*}\right)_{xx}$$
$$+K\left(q^{(s-1)} - 2q^{(s)} + q^{(s+1)}\right) + i\left[\lambda_s\left(\left|q^{(s)}\right|^2 q^{(s)}\right)_x\right.$$
$$\left.+\mu_s\left(\left|q^{(s)}\right|^2\right)_x q^{(s)} + \theta_s\left|q^{(s)}\right|^2\left(q^{(s)}\right)_x\right], \quad (42)$$

where $1 \le s \le M$ and K is the coupling coefficient. Eq. (42) represents the general model for optical couplers where coupling with nearest neighbors is considered. This model is decomposed by assuming:

$$q^{(s)}(x,t) = \Phi_s(\xi)e^{i(-\kappa x + \omega t + \varepsilon_0)}, \quad (43)$$

where Φ_s is the amplitude component of the soliton.

Substituting (43) and (4) into Eq. (42), the real part is given by:

$$b_s\Phi_s^{(4)} + 3\kappa(a_s - 2b_s\kappa)\Phi_s'' - (\beta_s + \gamma_s + 3\alpha_s)$$
$$\times \Phi_s^2\Phi_s'' - 6\alpha_s\Phi_s\Phi_s'^2 - (\omega + a_s\kappa^3 - b_s\kappa^4)\Phi_s$$

$$+[\kappa^2(\beta_s+\alpha_s+\gamma_s)+\kappa(\lambda_s+\theta_s)]\Phi_s^3$$
$$-K(\Phi_{s-1}-2\Phi_s+\Phi_{s+1})+c_s\Phi_s^{1-4n}+d_s\Phi_s^{1-3n}$$
$$+e_s\Phi_1^{1-2n}+f_s\Phi_s^{1-n}+g_s\Phi_s^{1+n}+h_s\Phi_s^{1+2n}$$
$$+k_s\Phi_s^{1+3n}+l_s\Phi_s^{1+4n}=0, \quad (44)$$

and the imaginary part is given by:

$$(a_s-4b_s\kappa)\Phi_s'''+[2\kappa(\beta_s-\gamma_s+3\alpha_s)+(3\lambda_s+2\mu_s$$
$$+\theta_s)]\Phi_s^2\Phi_s'-(v+3a_s\kappa^2-4b_s\kappa^3)\Phi_s'=0. \quad (45)$$

Eq. (45) gives the soliton frequency:

$$\kappa=\frac{a_s}{4b_s}, \quad (46)$$

the soliton velocity:

$$v=4b_s\kappa^3-3a_s\kappa^2, \quad (47)$$

and the constraint condition:

$$2\kappa(\beta_s-\gamma_s+3\alpha_s)$$
$$+(3\lambda_s+2\mu_s+\theta_s)=0. \quad (48)$$

Using the balancing principle in Eq. (44), one gets:

$$\Phi_{s-1}=\Phi_s=\Phi_{s+1}. \quad (49)$$

Consequently, Eq. (44) changes:

$$b_s\Phi_s^{(4)}+3\kappa(a_s-2b_s\kappa)\Phi_s''-(\beta_s+\gamma_s+3\alpha_s)$$
$$\times\Phi_s^2\Phi_s''-6\alpha_s\Phi_s\Phi_s'^2+[\kappa^2(\beta_s+\alpha_s+\gamma_s)$$
$$+\kappa(\lambda_s+\theta_s)]\Phi_s^3-(\omega+a_s\kappa^3-b_s\kappa^4)\Phi_s+c_s\Phi_s^{1-4n}$$
$$+d_s\Phi_s^{1-3n}+e_s\Phi_1^{1-2n}+f_s\Phi_s^{1-n}+g_s\Phi_s^{1+n}$$
$$+h_s\Phi_s^{1+2n}+k_s\Phi_s^{1+3n}+l_s\Phi_s^{1+4n}=0. \quad (50)$$

By using the transformation $\Phi_s(\xi)=Z^{\frac{1}{n}}(\xi)$, one has a new equation:

$$n^3b_sZ^3Z^{(4)}-4n^2(n-1)b_sZ^2Z'Z'''-3n^2$$
$$\times(n-1)b_sZ^2Z''^2+6n(2n-1)(n-1)b_sZZ'^2Z''$$
$$-(n-1)(2n-1)(3n-1)b_sZ'^4+3n^2\kappa$$
$$\times(a_s-2b_s\kappa)[nZ^3Z''-(n-1)Z^2Z'^2]+n^4c_s$$
$$+n^4d_sZ+n^4e_sZ^2+n^4f_sZ^3-n^4(\omega+a_s\kappa^3$$
$$-b_s\kappa^4)Z^4+n^4g_sZ^5+n^4h_sZ^6+n^4k_sZ^7+n^4l_sZ^8$$
$$+n^4[\kappa^2(\beta_s+\alpha_s+\gamma_s)+\kappa(\lambda_s+\theta_s)]Z^{\frac{2+4n}{n}}$$
$$-n^3(\beta_s+\gamma_s+3\alpha_s)Z^{\frac{2+3n}{n}}Z''+n^2[(n-1)$$
$$\times(\beta_s+\gamma_s)-3(n-3)\alpha_s]Z^{\frac{2+2n}{n}}Z'^2=0. \quad (51)$$

For integrability, one must select

$$(n-1)(\beta_s+\gamma_s)-3(n-3)\alpha_s=0,$$
$$\kappa(\beta_s+\alpha_s+\gamma_s)+\lambda_s+\theta_s=0,$$
$$\beta_s+\gamma_s+3\alpha_s=0. \quad (52)$$

Consequently, Eq. (51) changes to:

$$n^3b_sZ^3Z^{(4)}-4n^2(n-1)b_sZ^2Z'Z'''-3n^2$$
$$\times(n-1)b_sZ^2Z''^2+6n(2n-1)(n-1)b_sZZ'^2Z''$$
$$-(n-1)(2n-1)(3n-1)b_sZ'^4+3n^2\kappa(a_s-2b_s\kappa)$$
$$\times[nZ^3Z''-(n-1)Z^2Z'^2]+n^4c_s+n^4d_sZ+n^4e_sZ^2$$
$$+n^4f_sZ^3-n^4(\omega+a_s\kappa^3-b_s\kappa^4)Z^4$$
$$+n^4g_sZ^5+n^4h_sZ^6+n^4k_sZ^7+n^4l_sZ^8=0. \quad (53)$$

Now, we will solve Eq. (53) by using the following method:

3.1. New auxiliary equation approach

According to this method, balancing $Z^3Z^{(4)}$ and Z^8 in Eq. (53) by using (23), one gets $N=3$. It follows that Eq. (53) has the following solution form:

$$Z(\xi)=\Omega_0+\Omega_1F(\xi)+\Omega_2F^2(\xi)+\Omega_3F^3(\xi), \quad (54)$$

where $\Omega_0,\Omega_1,\Omega_2$ and Ω_3 are constants, provided $\Omega_3\neq 0$.

Family–1: Substituting (54) and (22) into Eq. (53), one gets algebraic equations. As a result, if we set $S_0=S_1=S_3=S_4=S_6=S_7=0$, in the obtained algebraic equations and solving it by using the Maple, one gets the following results:

$$\Omega_0=\frac{1}{2}\left[-\frac{9\kappa^2(6n^3+11n^2+6n+1)(a_s-2b_s\kappa)^2}{4(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4}},$$

$$\Omega_3=\left[-\frac{9\kappa^2(6n^3+11n^2+6n+1)(a_s-2b_s\kappa)^2}{4(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4}},$$

$$\Omega_1=\Omega_2=S_5=0,$$

$$S_2=\frac{n^2\kappa(a_s-2b_s\kappa)}{12(2n^2-2n+1)b_s},$$

$$S_8=-\frac{n^2\kappa(a_s-2b_s\kappa)}{6(2n^2-2n+1)b_s}, \quad (55)$$

and

$$\omega = -\frac{9\kappa^2(37n^2 - 32n + 9)(a_s - 2b_s\kappa)^2 + 32b_s(2n^2 - 2n + 1)^2(a_s\kappa^3 - b_s\kappa^4)}{32(2n^2 - 2n + 1)^2 b_s},$$

$$c_s = -\frac{81\kappa^4(36n^6 - 49n^4 + 14n^2 - 1)(a_s - 2b_s\kappa)^4}{4096(2n^2 - 2n + 1)^4 b_s^2 l_s},$$

$$f_s = \frac{9\kappa^2(3n^3 - 7n^2 + 3n - 2)(a_s - 2b_s\kappa)^2}{32(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4}},$$

$$g_s = -\frac{27n\kappa^2(n^2 - 4)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{-\frac{1}{4}},$$

$$h_s = -\frac{9\kappa^2(3n^3 + 19n^2 + 20n + 4)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{-\frac{1}{2}},$$

$$k_s = \frac{9\kappa^2(6n^3 + 13n^2 + 9n + 2)(a_s - 2b_s\kappa)^2}{2(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{-\frac{3}{4}},$$

$$d_s = 0, e_s = 0, \tag{56}$$

provided $b_s l_s < 0$. By using the solutions (24) and (25) one gets the following cases of solutions for Eq. (42):

Case–1: If $S_2 = \frac{n^2\kappa(a_s - 2b_s\kappa)}{12(2n^2 - 2n + 1)b_s} > 0$, then substituting (55) along with (24) into Eq. (54), one gets the bright soliton solutions as:

$$q^{(s)}(x,t) = \left[-\frac{9\kappa^2\left(6n^3 + 11n^2 + 6n + 1\right)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2} \pm \frac{\sqrt{2}}{2} \times \operatorname{sech}\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_s - 2b_s\kappa)}{(2n^2 - 2n + 1)b_s}} \times (x - vt) + \xi_{01}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x + \omega t + \varepsilon_0)}, \tag{57}$$

Case–2: If $S_2 = \frac{n^2\kappa(a_s - 2b_s\kappa)}{12(2n^2 - 2n + 1)b_s} < 0$, then substituting (55) along with (25) into Eq. (54), one gets the periodic solutions as:

$$q^{(s)}(x,t) = \left[-\frac{9\kappa^2\left(6n^3 + 11n^2 + 6n + 1\right)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2} \pm \frac{\sqrt{2}}{2} \times \csc\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_s - 2b_s\kappa)}{(2n^2 - 2n + 1)b_s}} \times (x - vt) + \xi_{02}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x + \omega t + \varepsilon_0)}, \tag{58}$$

The solutions (57) and (58) are satisfied under the constraint conditions (56).

Family–2: Substituting (54) and (22) into Eq. (53), one gets algebraic equations. As a result, if we set $S_0 = S_1 = S_3 = S_4 = S_6 = S_7 = 0$ and $S_8 = \frac{S_5^2}{4S_2}$, in the obtained algebraic equations and solving it by using the Maple, one gets the following results:

$$\Omega_0 = \frac{1}{2}\left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4}},$$

$$\Omega_3 = \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4}},$$

$$\Omega_1 = \Omega_2 = 0,$$

$$S_2 = \frac{n^2\kappa(a_s - 2b_s\kappa)}{3(2n^2 - 2n + 1)b_s},$$

$$S_5 = \frac{2n^2\kappa(a_s - 2b_s\kappa)}{3(2n^2 - 2n + 1)b_s}, \tag{59}$$

and

$$\omega = -\frac{9\kappa^2(3n^2 - 8n + 1)(a_s - 2b_s\kappa)^2 + 8b_s(2n^2 - 2n + 1)^2(a_s\kappa^3 - b_s\kappa^4)}{8b_s(2n^2 - 2n + 1)^2},$$

$$c_s = -\frac{81\kappa^4(36n^6 - 49n^4 + 14n^2 - 1)(a_s - 2b_s\kappa)^4}{256(2n^2 - 2n + 1)^4 b_s^2 l_s},$$

$$d_s = 0, e_s = 0, f_s = 0, g_s = 0, k_s = 0,$$

$$h_s = -\frac{12n\kappa(n+1)(a_s - 2b_s\kappa)}{(2n^2 - 2n + 1)}\sqrt{-\frac{l_s}{(6n^3 + 11n^2 + 6n + 1)b_s}}, \tag{60}$$

provided $b_s l_s < 0$. By using the solutions (26) and (27) one gets the following cases of solutions for Eq. (42):

Case–1: Substituting (59) along with (26) into Eq. (54), then one gets the dark soliton solutions as:

$$q^{(s)}(x,t) = \left[-\frac{9\kappa^2\left(6n^3+11n^2+6n+1\right)(a_s-2b_s\kappa)^2}{(2n^2-2n+1)^2 b_s l_s}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2}\tanh\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_s-2b_s\kappa)}{(2n^2-2n+1)b_s}} \times (x-vt)+ +\xi_{03}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x+\omega t+\varepsilon_0)}, \quad (61)$$

provided $(2n^2-2n+1)(a_s-2b_s\kappa)\kappa b_s > 0$.

Case–2: Substituting (59) along with (27) into Eq. (54), then one gets the singular soliton solutions as:

$$q^{(s)}(x,t) = \left[-\frac{9\kappa^2\left(6n^3+11n^2+6n+1\right)(a_s-2b_s\kappa)^2}{(2n^2-2n+1)^2 b_s l_s}\right]^{\frac{1}{4n}} \times \left\{\frac{1}{2}\coth\left[\frac{n}{2}\sqrt{\frac{3\kappa(a_s-2b_s\kappa)}{(2n^2-2n+1)b_s}} \times (x-vt)+ +\xi_{04}\right]\right\}^{\frac{1}{n}} \times e^{i(-\kappa x+\omega t+\varepsilon_0)}, \quad (62)$$

provided $(2n^2-2n+1)(a_s-2b_s\kappa)\kappa b_s > 0$.

The solutions (61) and (62) are satisfied under the constraint conditions (60).

4. Multiple–core couplers (Coupling with all neighbors)

The CQ system for all the existing neighbors in optical metamaterials having the generalized Kudryashov's law of arbitrary refractive index with perturbation terms is written as:

$$i\left(q^{(s)}\right)_t + ia_s\left(q^{(s)}\right)_{xxx} + b_s\left(q^{(s)}\right)_{xxxx} + \left[\frac{c_s}{|q^{(s)}|^{4n}} + \frac{d_s}{|q^{(s)}|^{3n}} + \frac{e_s}{|q^{(s)}|^{2n}} + \frac{f_s}{|q^{(s)}|^{n}} + g_s\left|q^{(s)}\right|^n + h_s\left|q^{(s)}\right|^{2n} + k_s\left|q^{(s)}\right|^{3n} + l_s\left|q^{(s)}\right|^{4n}\right]q^{(s)} = \alpha_s\left(\left|q^{(s)}\right|^2 q^{(s)}\right)_{xx} + \beta_s\left|q^{(s)}\right|^2\left(q^{(s)}\right)_{xx} + \gamma_s\left(q^{(s)}\right)^2\left(q^{(s)*}\right)_{xx} + \sum_{m=1}^{M}\chi_{sm}q^{(m)} + i\left[\lambda_s\left(\left|q^{(s)}\right|^2 q^{(s)}\right)_x + \mu_s\left(\left|q^{(s)}\right|^2\right)_x q^{(s)} + \theta_s\left|q^{(s)}\right|^2\left(q^{(s)}\right)_x\right], \quad (63)$$

where $1 \le s \le M$, while χ_{sm} represents the coupling coefficient with all neighbors. This model will be solved by taking the same assumption (43).

Substituting (43) and (4) into Eq. (63), give the same imaginary part (45), which recovers the same relations (46)-(48) and the real part as:

$$b_s\Phi_s^{(4)} + 3\kappa(a_s-2b_s\kappa)\Phi_s'' - (\beta_s+\gamma_s+3\alpha_s) \times \Phi_s^2\Phi_s'' - 6\alpha_s\Phi_s\Phi_s'^2 - (\omega+a_s\kappa^3-b_s\kappa^4)\Phi_s - \sum_{m=1}^{M}\chi_{sm}\Phi_m + [\kappa^2(\beta_s+\alpha_s+\gamma_s)+\kappa(\lambda_s+\theta_s)] \times \Phi_s^3 + c_s\Phi_s^{1-4n} + d_s\Phi_s^{1-3n} + e_s\Phi_1^{1-2n} + f_s\Phi_s^{1-n} + g_s\Phi_s^{1+n} + h_s\Phi_s^{1+2n} + k_s\Phi_s^{1+3n} + l_s\Phi_s^{1+4n} = 0. \quad (64)$$

Utilization the balancing principle induces the result:

$$\Phi_m = \Phi_s. \quad (65)$$

Consequently, Eq. (64) becomes:

$$b_s\Phi_s^{(4)} + 3\kappa(a_s-2b_s\kappa)\Phi_s'' - (\beta_s+\gamma_s+3\alpha_s) \times \Phi_s^2\Phi_s'' - 6\alpha_s\Phi_s\Phi_s'^2 - (\omega+a_s\kappa^3-b_s\kappa^4 + \sum_{m=1}^{M}\chi_{sm})\Phi_s + [\kappa^2(\beta_s+\alpha_s+\gamma_s)+\kappa(\lambda_s+\theta_s)] \times \Phi_s^3 + c_s\Phi_s^{1-4n} + d_s\Phi_s^{1-3n} + e_s\Phi_1^{1-2n} + f_s\Phi_s^{1-n} + g_s\Phi_s^{1+n} + h_s\Phi_s^{1+2n} + k_s\Phi_s^{1+3n} + l_s\Phi_s^{1+4n} = 0. \quad (66)$$

Using the transformation $\Phi_s(\xi) = H^{\frac{1}{n}}(\xi)$, gives a new equation:

$$n^3 b_s H^3 H^{(4)} - 4n^2(n-1)b_s H^2 H' H''' - 3n^2(n-1) \times b_s H^2 H''^2 + 6n(2n-1)(n-1)b_s H H'^2 H'' - (n-1) \times (2n-1)(3n-1)b_s H'^4 + 3n^2\kappa(a_s-2b_s\kappa) \times [nH^3H'' - (n-1)H^2H'^2] + n^4 c_s + n^4 d_s H + n^4 e_s H^2 + n^4 f_s H^3 - n^4(\omega + a_s\kappa^3 - b_s\kappa^4 + \sum_{m=1}^{M}\chi_{sm})H^4$$

$$+n^4 g_s H^5 + n^4 h_s H^6 + n^4 k_s H^7 + n^4 l_s H^8 + n^4[\kappa^2(\beta_s + \alpha_s$$

$$+\gamma_s) + \kappa(\lambda_s + \theta_s)]H^{\frac{2+4n}{n}} - n^3(\beta_s + \gamma_s + 3\alpha_s)H^{\frac{2+3n}{n}}H''$$

$$+n^2[(n-1)(\beta_s + \gamma_s) - 3(n-3)\alpha_s]H^{\frac{2+2n}{n}}H'^2 = 0. \tag{67}$$

For integrability, one must select

$$(3n-2)(\beta_s + \gamma_s) - 9(n-2)\alpha_s = 0,$$

$$\kappa(\beta_s + \alpha_s + \gamma_s) + \lambda_s + \theta_s = 0,$$

$$\beta_s + \gamma_s + 3\alpha_s = 0,$$

$$c_s = 0,\ d_s = 0,\ f_s = 0,\ g_s = 0. \tag{68}$$

Consequently, Eq. (67) changes to:

$$n^3 b_s H^3 H^{(4)} - 4n^2(n-1)b_s H^2 H' H''' - 3n^2 \times (n-1)b_s H^2 H''^2 + 6n(2n-1)(n-1)b_s H H'^2 \times H'' - (n-1)(2n-1)(3n-1)b_s H'^4 + 3n^2\kappa \times (a_s - 2b_s\kappa)[nH^3H'' - (n-1)H^2H'^2] + n^4 c_s$$

$$+n^4 d_s H + n^4 e_s H^2 + n^4 f_s H^3 - n^4(\omega + a_s\kappa^3 - b_s\kappa^4 + \sum_{m=1}^{M} \chi_{sm})H^4 + n^4 g_s H^5$$

$$+n^4 h_s H^6 + n^4 k_s H^7 + n^4 l_s H^8 = 0. \tag{69}$$

Next, we will solve Eq. (69) by using the following method:

4.1. New auxiliary equation approach

According to this method, balancing $H^3H^{(4)}$ and H^8 in Eq. (69) by using (23), one gets $N = 3$. It follows that Eq. (69) has the following solution form:

$$H(\xi) = \Omega_0 + \Omega_1 F(\xi) + \Omega_2 F^2(\xi) + \Omega_3 F^3(\xi), \tag{70}$$

where $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 are constants, provided $\Omega_3 \neq 0$.

Family–1: Substituting (70) and (22) into Eq. (69), one gets algebraic equations. As a result, if we set $S_0 = S_1 = S_3 = S_4 = S_6 = S_7 = 0$, in the obtained algebraic equations and solving it by using the Maple, one gets the following results:

$$\Omega_0 = \frac{1}{2}\left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4}},$$

$$\Omega_3 = \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4}},$$

$$S_2 = \frac{n^2\kappa(a_s - 2b_s\kappa)}{12(2n^2 - 2n + 1)b_s},$$

$$S_8 = -\frac{n^2\kappa(a_s - 2b_s\kappa)}{6(2n^2 - 2n + 1)b_s},$$

$$\Omega_1 = \Omega_2 = S_5 = 0, \tag{71}$$

and

$$\omega = -\frac{9\kappa^2(37n^2 - 32n + 9)(a_s - 2b_s\kappa)^2 + 32b_s(2n^2 - 2n + 1)^2\left(a_s\kappa^3 - b_s\kappa^4 + \sum_{m=1}^{M}\chi_{sm}\right)}{32(2n^2 - 2n + 1)^2 b_s},$$

$$c_s = -\frac{81\kappa^4(36n^6 - 49n^4 + 14n^2 - 1)(a_s - 2b_s\kappa)^4}{4096(2n^2 - 2n + 1)^4 b_s^2 l_s},$$

$$f_s = \frac{9\kappa^2(3n^3 - 7n^2 + 3n - 2)(a_s - 2b_s\kappa)^2}{32(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{\frac{1}{4}},$$

$$g_s = -\frac{27n\kappa^2(n^2 - 4)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{-\frac{1}{4}},$$

$$h_s = -\frac{9\kappa^2(3n^3 + 19n^2 + 20n + 4)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{-\frac{1}{2}},$$

$$k_s = \frac{9\kappa^2(6n^3 + 13n^2 + 9n + 2)(a_s - 2b_s\kappa)^2}{2(2n^2 - 2n + 1)^2 b_s} \times \left[-\frac{9\kappa^2(6n^3 + 11n^2 + 6n + 1)(a_s - 2b_s\kappa)^2}{4(2n^2 - 2n + 1)^2 b_s l_s}\right]^{-\frac{3}{4}},$$

$$d_s = 0,\ e_s = 0, \tag{72}$$

provided $b_s l_s < 0$. By using the solutions (24) and (25) one gets the following cases of solutions for Eq. (63):

Case–1: If $S_2 = \frac{n^2\kappa(a_s - 2b_s\kappa)}{12(2n^2 - 2n + 1)b_s} > 0$, then substituting (71) along with (24) into Eq. (70), one gets the bright soliton solutions as:

$$q^{(s)}(x,t)=\left[-\frac{9\kappa^2\begin{pmatrix}6n^3+11n^2\\+6n+1\end{pmatrix}(a_s-2b_s\kappa)^2}{4(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4n}}$$

$$\times\left\{\begin{matrix}\frac{1}{2}\pm\frac{\sqrt{2}}{2}\\ \times\operatorname{sech}\left[\begin{matrix}\frac{n}{2}\sqrt{\frac{3\kappa(a_s-2b_s\kappa)}{(2n^2-2n+1)b_s}}\\ \times(x-vt)+\xi_{01}\end{matrix}\right]\end{matrix}\right\}^{\frac{1}{n}}$$

$$\times e^{i(-\kappa x+\omega t+\varepsilon_0)},\qquad(73)$$

Case–2: If $S_2=\frac{n^2\kappa(a_s-2b_s\kappa)}{12(2n^2-2n+1)b_s}<0$, then substituting (71) along with (25) into Eq. (70), one gets the periodic solutions as:

$$q^{(s)}(x,t)=\left[-\frac{9\kappa^2\begin{pmatrix}6n^3+11n^2\\+6n+1\end{pmatrix}(a_s-2b_s\kappa)^2}{4(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4n}}$$

$$\times\left\{\begin{matrix}\frac{1}{2}\pm\frac{\sqrt{2}}{2}\\ \times\csc\left[\begin{matrix}\frac{n}{2}\sqrt{\frac{3\kappa(a_s-2b_s\kappa)}{(2n^2-2n+1)b_s}}\\ \times(x-vt)+\xi_{02}\end{matrix}\right]\end{matrix}\right\}^{\frac{1}{n}}$$

$$\times e^{i(-\kappa x+\omega t+\varepsilon_0)}.\qquad(74)$$

The solutions (73) and (74) are satisfied under the constraint conditions (72).

Family–2: Substituting (70) and (22) into Eq. (69), one gets algebraic equations. As a result, if we set $S_0=S_1=S_3=S_4=S_6=S_7=0$ and $S_8=\frac{S_5^2}{4S_2}$, in the obtained algebraic equations and solving it by using the Maple, one gets the following results:

$$\Omega_0=\frac{1}{2}\left[-\frac{9\kappa^2\begin{pmatrix}6n^3+11n^2\\+6n+1\end{pmatrix}(a_s-2b_s\kappa)^2}{(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4}},$$

$$\Omega_3=\left[-\frac{9\kappa^2\begin{pmatrix}6n^3+11n^2\\+6n+1\end{pmatrix}(a_s-2b_s\kappa)^2}{(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4}},$$

$$S_2=\frac{n^2\kappa(a_s-2b_s\kappa)}{3(2n^2-2n+1)b_s},$$

$$S_5=\frac{2n^2\kappa(a_s-2b_s\kappa)}{3(2n^2-2n+1)b_s},$$

$$\Omega_1=\Omega_2=0,\qquad(75)$$

and

$$\omega=-\frac{\begin{matrix}9\kappa^2(3n^2-8n+1)(a_s-2b_s\kappa)^2\\ +8b_s(2n^2-2n+1)^2\begin{pmatrix}a_s\kappa^3-b_s\kappa^4\\+\sum_{m=1}^{M}\chi_{sm}\end{pmatrix}\end{matrix}}{8b_s(2n^2-2n+1)^2},$$

$$c_s=-\frac{81\kappa^4\begin{pmatrix}36n^6-49n^4\\+14n^2-1\end{pmatrix}(a_s-2b_s\kappa)^4}{256(2n^2-2n+1)^4b_s^2l_s},$$

$$h_s=-\frac{12n\kappa(n+1)(a_s-2b_s\kappa)}{(2n^2-2n+1)}$$

$$\times\sqrt{-\frac{l_s}{(6n^3+11n^2+6n+1)b_s}},$$

$$d_s=0,\ e_s=0,\ f_s=0,\ g_s=0,\ k_s=0,\qquad(76)$$

provided $b_sl_s<0$. By using the solutions (26) and (27) one gets the following cases of solutions for Eq. (63):

Case–1: Substituting (75) along with (26) into Eq. (70), then one gets the dark soliton solutions as:

$$q^{(s)}(x,t)=\left[-\frac{9\kappa^2\begin{pmatrix}6n^3+11n^2\\+6n+1\end{pmatrix}(a_s-2b_s\kappa)^2}{(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4n}}$$

$$\times\left\{\frac{1}{2}\tanh\left[\begin{matrix}\frac{n}{2}\sqrt{\frac{3\kappa(a_s-2b_s\kappa)}{(2n^2-2n+1)b_s}}\\ \times(x-vt)++\xi_{03}\end{matrix}\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i(-\kappa x+\omega t+\varepsilon_0)},\qquad(77)$$

provided $(2n^2-2n+1)(a_s-2b_s\kappa)\kappa b_s>0$.

Case–2: Substituting (75) along with (27) into Eq. (70), then one gets the singular soliton solutions as:

$$q^{(s)}(x,t)=\left[-\frac{9\kappa^2\begin{pmatrix}6n^3+11n^2\\+6n+1\end{pmatrix}(a_s-2b_s\kappa)^2}{(2n^2-2n+1)^2b_sl_s}\right]^{\frac{1}{4n}}$$

$$\times\left\{\frac{1}{2}\coth\left[\begin{matrix}\frac{n}{2}\sqrt{\frac{3\kappa(a_s-2b_s\kappa)}{(2n^2-2n+1)b_s}}\\ \times(x-vt)++\xi_{04}\end{matrix}\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i(-\kappa x+\omega t+\varepsilon_0)},\qquad(78)$$

provided $(2n^2 - 2n + 1)(a_s - 2b_s\kappa)\kappa b_s > 0$.

The solutions (77) and (78) stay valid under the constraint conditions (76).

5. Conclusions

Today's paper is about the isolation of soliton solutions to the governing model that is with CQ dispersive effect coupled with Kudryashov's generalized form of SPM. Twin–core couplers along with multiple–core couplers are studied in this paper. The new auxiliary equation approach identifies the full spectrum of soliton solutions that are presented. Thus, a new chapter from optical couplers have opened up that gave way to a lot of untraversed open avenues. An immediate thought would be to identify the conservation laws. Other avenues to walk through are looking into the recovery of soliton parameter dynamics by the application of the variational principle or moment method or even with the application of collective variables. Next, the quasi-stationarity as well as quasimonochromaticity effects are also to be touched base on. These would keep the authors busy !

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