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Dark and singular optical solitons with spatio-temporal dispersion by Kudryashov's method

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This paper obtains exact 1-soliton solution to the nonlinear Schrödinger's equa5tion with spatio-temporal dispersion, in addition to usual group-velocity dispersion, by the aid of Kudryashov's method. There are four types of nonlinearity that are studied. These are Kerr law, power law, parabolic law and dual-power law. Bright, dark and singular soliton solutions are obtained.

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1. Introduction

Optical soliton molecules form the basic fabric in communication industry for trans-continental and trans-oceanic distances. These solitons are the outcome of a delicate balance between dispersion and nonlinearity [1-20]. The basic model that describes this engineering dynamics is the nonlinear Schrödinger's equation (NLSE). This paper considers NLSE with spatio-temporal dispersion (STD) in addition to the usual group-velocity dispersion (GVD). The inclusion of STD makes the problem of optical soliton transmission well-posed as introduced in 2012 [6]. Additionally, STD addresses the aspect of Internet bottleneck that is a growing problem during current time due to a surge in Internet activity.

Integrability of NLSE is one of the basic issues that need to be addressed. Obtaining an exact 1-soliton solution is imperative for NLSE. These exact soliton solutions play an important role in communications industry. This paper is thus going to retrieve exact 1-soliton solutions to the governing NLSE with four forms of nonlinear media. These are Kerr law, power law, parabolic law and dual-power law nonlinearity. The integration technique that is implemented in this paper is known as Kudryashov's method. Bright, dark and singular soliton solutions are obtained for these forms of nonlinear media.

2. Governing equation

The dimensionless form of NLSE with STD is given by [7, 8]

$$i q_t + a q_{tx} + b q_{xx} + c F\left(|q|^2\right) q = 0 \tag{1}$$

where x represents the non-dimensional distance along the fiber while, t represents temporal variable in dimensionless form and a and b are real valued constants. The dependent variable $q(x,t)$ is a complex valued soliton profile. On the left side of this equation the first term represents the evolution term, while the coefficient of a is STD and coefficient of b is GVD. It is the inclusion of this STD makes the governing NLSE well-posed as was indicated a couple of years ago [6]. Also, c is the coefficient of the nonlinear term where the functional F represents the non-Kerr law nonlinearity in general. Solitons are the outcome of a delicate balance between dispersion and nonlinearity from equation (1).

In Eq. (1), F is a real-valued algebraic function and one needs to have smoothness of the complex-valued function $F\left(|q|^2\right) q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R_2, the function $F\left(|q|^2\right)q$ is k times continuously differentiable, and consequently

$$F\left(|q|^2\right)q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m);R^2\right). \tag{2}$$

3. Soliton solutions

In order to solve Eq. (1) by the Q-function method [5], we use the following wave transformation

$$q(x,t)=U(\xi)e^{i\Phi(x,t)} \tag{3}$$

where $U(\xi)$ represents the shape of the pulse and

$$\xi=\mu\left(x-vt\right), \tag{4}$$

$$\Phi\left(x,t\right)=-\kappa x+\omega t+\theta. \tag{5}$$

In Eq. (3), the function $\Phi\left(x,t\right)$ represents the phase component of the soliton. Then, in Eq. (5), κ is the soliton frequency, while ω is the wave number of the soliton and θ is the phase constant. Finally, in Eq. (4), v is the velocity of the soliton. The definition of the parameter μ is based on the type of soliton considered. For bright solitons, it represents inverse width, while for dark and singular solitons, it represents a free parameter. Substituting Eq. (3) into Eq. (1) and then decomposing into real and imaginary parts yields a pair of relations. The imaginary part gives

$$v=\frac{a\omega-2b\kappa}{1-a\kappa}, \tag{6}$$

while the real part gives

$$\mu^2\left(b-av\right)U''-\left(\omega-a\omega\kappa+b\kappa^2\right)U+cF\left(U^2\right)U=0. \tag{7}$$

Here U'' represents $\frac{d^2U}{d\xi^2}$. The relation (6) gives the velocity of the soliton in terms of the wave number while Eq. (7) can be integrated to compute the soliton profile provided the functional F is given.

The improved NLSE, that is with STD, will be considered for the following four forms of nonlinearity as discussed in the following four subsections. The aim of this paper is to find exact solution of the Eq. (1) by using Kudryashov's method [5].

3.1. Kerr law

The Kerr law of nonlinearity originates from the fact that a light wave in an optical fiber faces nonlinear responses from non-harmonic motion of electrons bound in molecules, caused by an external electric field. Even though the nonlinear responses are extremely weak, their effects appear in various ways over long distance of propagation that is measured in terms of light wavelength. The origin of nonlinear response is related to the nonharmonic motion of bound electrons under the influence of an applied field. As a result the induced polarization is not linear in the electric field, but involves higher order terms in electric field amplitude.

The Kerr law nonlinearity is the case when $F(u)=u,$ so that Eq. (1) reduces to [7-10]

$$iq_t+aq_{tx}+bq_{xx}+c|q|^2q=0 \tag{8}$$

and Eq. (7) simplifies to

$$\mu^2\left(b-av\right)U''-\left(\omega-a\omega\kappa+b\kappa^2\right)U+cU^3=0. \tag{9}$$

By means of the Q function method, we can look for exact solutions of Eq. (9) in the form of the following power series [5]

$$U(\xi)=\sum_{l=0}^{M}A_l\,Q^l(\xi),\ \ Q(\xi)=\frac{1}{1+e^{-\xi-\xi_0}}, \tag{10}$$

where M is a positive integer, in most cases, that will be determined and ξ_0 is an arbitrary constant. To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms.

One can see that the function $Q(\xi)$ is solution of the equation

$$Q_\xi=Q-Q^2. \tag{11}$$

Equation (11) allows us to obtain U' and U'' using polynomials of Q. The balancing procedure yields $M=1$. Thus, to look for the solution of Eq. (9) we can use following formulae

$$U(\xi)=A_0+A_1\,Q(\xi). \tag{12}$$

Substituting Eqs. (12) into Eq. (9), collecting all terms with the same powers of $Q(\xi)$ and setting each coefficient to zero, we obtain a system of algebraic equations for $A_0, A_1, \omega, \kappa, v$ and μ as follows.

$Q^3(\xi)$ coeff.:

$$2\mu^2\left(b-av\right)A_1+cA_1^3=0,$$

$Q^2(\xi)$ coeff.:

$$3\mu^2\left(b-av\right)A_1+3cA_0A_1^2=0,$$

$Q^1(\xi)$ coeff.:

$$A_1\left(b-av\right)\mu^2+3cA_0^2A_1-\left(\omega-a\omega\kappa+b\kappa^2\right)A_1=0, \tag{13}$$

$Q^0(\xi)$ coeff.:

$$cA_0^3-\left(\omega-a\omega\kappa+b\kappa^2\right)A_0=0.$$

Solving this system with the aid of Maple gives

$$A_0=\pm\sqrt{\frac{b\kappa^2+\omega-a\omega\kappa}{c}},\quad A_1=\mp2\sqrt{\frac{b\kappa^2+\omega-a\omega\kappa}{c}},\quad \mu=\pm\sqrt{\frac{2(b\kappa^2+\omega-a\omega\kappa)}{av-b}}, \tag{14}$$

where κ and ω are arbitrary constants. Substituting the solution set (14) into Eq. (12), the exact solutions of Eq. (8) can be written as:

Type-1: When $\left(\omega-a\omega\kappa-b\kappa^2\right)\left(av-b\right)>0$, we have

(1) Dark 1-soliton solution:

$$q_{1,2}(x,t)=\pm\sqrt{\frac{\omega-a\omega\kappa+b\kappa^2}{c}}\tanh\left[\sqrt{\frac{\omega-a\omega\kappa+b\kappa^2}{2\left(av-b\right)}}\left(x-vt\right)\right]e^{i\left(-\kappa x+\omega t+\theta\right)} \tag{15}$$

(2) Singular 1-soliton solution:

$$q_{3,4}(x,t)=\pm\sqrt{\frac{\omega-a\omega\kappa+b\kappa^2}{c}}\coth\left[\sqrt{\frac{\omega-a\omega\kappa+b\kappa^2}{2\left(av-b\right)}}\left(x-vt\right)\right]e^{i\left(-\kappa x+\omega t+\theta\right)} \tag{16}$$

Type-2: When $\left(\omega-a\omega\kappa-b\kappa^2\right)\left(av-b\right)<0$, we have the following singular periodic solutions

$$q_{5,6}(x,t)=\pm\sqrt{\frac{a\omega\kappa-\omega-b\kappa^2}{c}}\tan\left[\sqrt{\frac{a\omega\kappa-\omega-b\kappa^2}{2\left(av-b\right)}}\left(x-vt\right)\right]e^{i\left(-\kappa x+\omega t+\theta\right)} \tag{17}$$

$$q_{7,8}(x,t)=\mp\sqrt{\frac{a\omega\kappa-\omega-b\kappa^2}{c}}\cot\left[\sqrt{\frac{a\omega\kappa-\omega-b\kappa^2}{2\left(av-b\right)}}\left(x-vt\right)\right]e^{i\left(-\kappa x+\omega t+\theta\right)} \tag{18}$$

It needs to be noted that the singular periodic solutions that are listed in Type-2 are not applicable in optics. However, these solutions that appear as a by-product are listed for completeness.

3.2. Power law

Power law nonlinearity is exhibited in various materials including semiconductors and is also viewed as a generalized version of Kerr law nonlinear medium. In this case [7-10]

$$F(u)=u^n \tag{19}$$

so that Eq. (1) collapses to

$$iq_t+aq_{tx}+bq_{xx}+c\left|q\right|^{2n}q=0. \tag{20}$$

Here in Eq. (20) the parameter n dictates the power law nonlinearity. For stability issues, one needs

$$0<n<2 \tag{21}$$

and in particular $n\neq2$ to circumvent self-focusing singularity.

In this subsection, we would like to extend the functional variable method to solve the improved NLSE with power law nonlinearity. Substituting Eq. (3) into Eq. (20) and then decomposing into real and imaginary parts yields a pair of relations. The imaginary part gives (6), and real part leads to

$$\mu^2\left(b-av\right)U''-\left(\omega-a\omega\kappa+b\kappa^2\right)U+cU^{2n+1}=0. \tag{22}$$

According to the previous steps, using the balancing procedure between U'' and U^{2n+1} in Eq. (22) we get

$$N+2=(2n+1)N\Leftrightarrow 2nN=2\Leftrightarrow N=\frac{1}{n}.$$

To obtain an analytic solution, we use the transformation $U=V^{\frac{1}{2n}}$ in Eq. (22) to find

$$\mu^2\left(b-av\right)\left\{(1-2n)\left(V'\right)^2+2nVV''\right\}-4\left(\omega-a\omega\kappa+b\kappa^2\right)n^2V^2+4cn^2V^3=0. \tag{23}$$

Suppose that the solutions of Eq. (23) can be expressed by a polynomial in $Q(\xi)$ as follows:

$$V(\xi)=\sum_{l=0}^{M}A_l\,Q^l(\xi), \tag{24}$$

where a_l are real constants with $A_M\neq 0$ and M is a positive integer which can be determined by balancing the highest order derivative term with the highest order nonlinear term after substituting ansatz (24) into Eq. (23), where $Q(\xi)$ satisfies Eq. (11).

Balancing the order of VV'' and V^3 in Eq. (23), we have $N=2$. Therefore; Eq. (23) can be rewritten as

$$V(\xi)=A_0+A_1\,Q(\xi)+A_2\,Q^2(\xi). \tag{25}$$

Substituting Eq. (25) in Eq. (23) and setting all the coefficients of powers $Q(\xi)$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\begin{aligned}
A_0&=0,\\
A_1&=\frac{4\mu^2(n+1)b}{c(-8n^2a\kappa+\mu^2a^2+4n^2+4n^2a^2\kappa^2)},\\
A_2&=-\frac{4\mu^2(n+1)b}{c(-8n^2a\kappa+\mu^2a^2+4n^2+4n^2a^2\kappa^2)},\\
\omega&=\frac{b\left(\mu^2-4n^2\kappa^2+4n^2\kappa^3a+\mu^2a\kappa\right)}{4n^2-8n^2a\kappa+\mu^2a^2+4n^2a^2\kappa^2}.
\end{aligned} \tag{26}$$

where κ and μ are arbitrary constants.

Substituting the solution set (26) into (25), the solution formulae of Eq. (23) can be written as

$$V(\xi)=\frac{4\mu^2(n+1)b}{c(-8n^2a\kappa+\mu^2a^2+4n^2+4n^2a^2\kappa^2)}\left(Q(\xi)-Q^2(\xi)\right). \tag{27}$$

Using the transformation $U=V^{\frac{1}{2n}}$, we can obtain the following exact solution of Eq. (20):

$$q(x,t)=\left[\frac{\mu^2(n+1)b}{c(-8n^2a\kappa+\mu^2a^2+4n^2+4n^2a^2\kappa^2)}sech^2\left\{\frac{\mu}{2}\left(x-vt\right)\right\}\right]^{\frac{1}{2n}}e^{i(-\kappa x+\omega t+\theta)}, \tag{28}$$

where ω is given by (26).

3.3. Parabolic law

For parabolic law nonlinearity,

$$F(u)=c_1u+c_2u^2 \tag{29}$$

where c_1 and c_2 are constants. Therefore, Eq. (1) takes the form [2, 7-12]

$$iq_t+aq_{tx}+bq_{xx}+\left(c_1|q|^2+c_2|q|^4\right)q=0. \tag{30}$$

In this case, Eq. (7) simplifies to

$$\mu^2\left(b-av\right)U''-\left(\omega-a\,\omega\kappa+b\,\kappa^2\right)U+c_1U^3+c_2U^5=0. \tag{31}$$

Balancing U'' with U^5 in Eq. (31) we have

$$M+2=5M\Leftrightarrow 2=4M\Leftrightarrow M=\frac{1}{2}.$$

To obtain an analytic solution, we use the transformation $U=V^{\frac{1}{2}}$ in Eq. (31) to find

$$\mu^2\left(b-av\right)\left\{2VV''-\left(V'\right)^2\right\}-4\left(\omega-a\omega\kappa+b\kappa^2\right)V^2+4c_1V^3+4c_2V^4=0. \tag{32}$$

Balancing VV'' with V^4 in Eq. (32) we find that $M=1$. This implies

$$V(\xi)=A_0+A_1Q(\xi). \tag{33}$$

Substituting Eq. (33) in Eq. (32) and setting all the coefficients of powers $Q(\xi)$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

Case-I:

$$A_0=0,\ A_1=-\frac{3c_1}{4c_2},\ \mu=\pm\frac{3c_1}{2\sqrt{3c_2(va-b)}},$$
$$\omega=\frac{3}{16}\frac{c_1^2}{c_2(a\kappa-1)}+\frac{b\kappa^2}{a\kappa-1}. \tag{34}$$

Substituting the solution set (34) into Eq. (33), a kind of kink-type solitary solution of Eq. (30) can be written as

$$q(x,t)=\left\{-\frac{3c_1}{8c_2}\left[1+\tanh\left(\pm\frac{3c_1}{4\sqrt{3c_2(av-b)}}(x-vt)\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \tag{35}$$

where

$$v=\frac{a\left\{\frac{3}{16}\frac{c_1^2}{c_2(a\kappa-1)}+\frac{b\kappa^2}{a\kappa-1}\right\}-2b\kappa}{1-a\kappa}. \tag{36}$$

and

$$\omega=\frac{3}{16}\frac{c_1^2}{c_2(a\kappa-1)}+\frac{b\kappa^2}{a\kappa-1} \tag{37}$$

Case-II:

$$A_0=-\frac{3c_1}{4c_2},\ A_1=\frac{3c_1}{4c_2},\ \mu=\pm\frac{3c_1}{2\sqrt{3c_2(va-b)}},$$
$$\omega=\frac{3}{16}\frac{c_1^2}{c_2(a\kappa-1)}+\frac{b\kappa^2}{a\kappa-1}. \tag{38}$$

Substituting the solution set (38) into Eq. (33), a topological soliton solution of Eq. (30) can be written as

$$q(x,t)=\left\{-\frac{3c_1}{8c_2}\left[1-\tanh\left(\pm\frac{3c_1}{4\sqrt{3c_2(av-b)}}(x-vt)\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \tag{39}$$

with v and ω given by (36) and (37) respectively. The solutions (35) and (39) are indeed dark soliton solutions to the governing NLSE with STD.

3.4. Dual-power law

This model describes solitons in photovoltaic-photo refractive materials such as $LiNbO_3$. For dual-power law nonlinearity [2, 7-12]

$$F(u)=c_1u^n+c_2u^{2n} \tag{40}$$

where c_1 and c_2 are constants. Therefore, Eq. (1) reduces to

$$iq_t+aq_{tx}+bq_{xx}+\left(c_1|q|^{2n}+c_2|q|^{4n}\right)q=0 \tag{41}$$

so that, for $n=1$, dual-power law collapses to the parabolic law. In this case, Eq. (7) simplifies to

$$\mu^2(b-av)U''-\left(\omega-a\omega\kappa+b\kappa^2\right)U+c_1U^{2n+1}+c_2U^{4n+1}=0. \tag{42}$$

Balancing U'' with U^{4n+1} in Eq. (42) we have

$$M+2=(4n+1)M\Leftrightarrow 2=4nM\Leftrightarrow M=\frac{1}{2n}.$$

To obtain an analytic solution, we propose a transformation denoted by $U=V^{\frac{1}{2n}}$. Then Eq. (42) is converted to

$$\mu^2(b-av)\left\{2nVV''+(1-2n)(V')^2\right\}-4n^2\left(\omega-a\omega\kappa+b\kappa^2\right)V^2+4n^2c_1V^3+4n^2c_2V^4=0. \tag{43}$$

Balancing VV'' with V^4 in Eq. (43) we find that $M=1$. This means

$$V(\xi)=A_0+A_1Q(\xi). \tag{44}$$

Substituting Eq. (44) in Eq. (43) and setting all the coefficients of powers $Q(\xi)$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

Case-I:

$$A_0 = 0,\ A_1 = -\frac{(1+2n)c_1}{2(1+n)c_2},\ \mu = \pm\frac{nc_1}{n+1}\sqrt{\frac{1+2n}{c_2(av-b)}},$$
$$\omega = \frac{(1+2n)c_1^2}{4(1+n)^2 c_2(a\kappa-1)} + \frac{b\kappa^2}{a\kappa-1}. \tag{45}$$

Substituting the solution set (45) into Eq. (44), a kind of kink-type solitary solution of Eq. (41) can be written as

$$q(x,t) = \left\{ -\frac{(1+2n)c_1}{4(1+n)c_2}\left[1+\tanh\left(\pm\frac{nc_1}{2(n+1)}\sqrt{\frac{1+2n}{c_2(av-b)}}\left(x-vt\right)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{46}$$

where

$$v = \frac{a\left\{\frac{(1+2n)c_1^2}{4(1+n)^2 c_2(a\kappa-1)} + \frac{b\kappa^2}{a\kappa-1}\right\} - 2b\kappa}{1-a\kappa}. \tag{47}$$

and

$$\omega = \frac{(1+2n)c_1^2}{4(1+n)^2 c_2(a\kappa-1)} + \frac{b\kappa^2}{a\kappa-1}. \tag{48}$$

Case-II:

$$A_0 = -\frac{(1+2n)c_1}{2(1+n)c_2},\ A_1 = \frac{(1+2n)c_1}{2(1+n)c_2},$$
$$\mu = \pm\frac{nc_1}{n+1}\sqrt{\frac{1+2n}{c_2(av-b)}}, \tag{49}$$
$$\omega = \frac{(1+2n)c_1^2}{4(1+n)^2 c_2(a\kappa-1)} + \frac{b\kappa^2}{a\kappa-1}.$$

Substituting the solution set (49) into Eq. (44), a kind of kink-type solitary solution of Eq. (41) can be written as

$$q(x,t) = \left\{ -\frac{(1+2n)c_1}{4(1+n)c_2}\left[1-\tanh\left(\left(\pm\frac{nc_1}{2(n+1)}\right)\left(\sqrt{\frac{1+2n}{c_2(av-b)}}\left(x-vt\right)\right)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{50}$$

where v and ω are defined in (47) and (48) respectively. The solutions represented by (46) and (50) are dark 1-soliton solutions to NLSE with STD in dual-power law medium. Finally, it is important to note that the soliton solutions given by (35), (39), (46) and (50) will exist for

$$c_1 c_2 < 0$$

and

$$c_2(av-b) > 0.$$

4. Conclusions

This paper addressed the soliton solutions by the aid of Kudryashov's method. This is yet another integration algorithm that is applied to NLSE to retrieve soliton solutions. This method obtains bright, dark and singular soliton solutions to the NLSE that is considered with STD in addition to GVD. It is interesting to observe that Kerr law gives dark and singular solitons only, while power law nonlinearity leads to a bright soliton solution. Therefore bright soliton solution can be retrieved from power law nonlinearity after setting the power law nonlinearity parameter $n=1$. Thus for Kudryashov's method, retrieving a bright 1-soliton solution for Kerr law nonlinearity is an indirect route. Then, for parabolic and dual-power laws of nonlinearity, it is only dark soliton solution that is retrievable, which gives the limitation of this approach.

The results of this paper stand with a lot of promise in future. Later, soliton solutions of NLSE with perturbation terms will be obtained by the aid of Kudryashov's method. Additionally, this integration architecture will be implemented to study optical couplers as well as birefringent fibers and DWDM systems. These form a tip of the iceberg. The results of those research will be reported in due time.

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