

Exact solutions of the perturbed modified KdV equation using the functional variable method

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In this paper, new travelling wave solutions in hyperbolic function form and trigonometric function form of the perturbed modified KdV equation is successfully found out by using the functional variable method. Due to the good performance of the functional variable method, it is believed that this method is a promising technique in handling a wide variety of partial differential equations. We checked the correctness of the obtained results by putting them back into the original equation with the aid of MAPLE. This provides an extra measure of confidence in the results.

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1. Introduction

Several powerful methods have been proposed to obtain exact solutions of nonlinear partial differential equations, such as the modified simple equation method [1-54], the extended trial equation method [3-4], the sine-cosine function method [5-6], the G'/G -expansion Method [7-8], the first integral method [9-11], the exp-function method [12], the Riccati sub method [13-14], the modified Kudryashov method [15] and so on [16-37].

It is well known that the modified Korteweg de Vries (MKdV) equation is an important nonlinear evolution equation, since it arises in applications to the dynamics of thin elastic rods [16], phonons in anharmonic lattices [17], meandering ocean jets [18], traffic congestion [19-20], hyperbolic surfaces [21], and ion acoustic solitons [22]. The general form of MKdV equation is

$$u_t + au^2u_x + \gamma u_{xxx} = 0, \gamma > 0 \quad (1)$$

where u represents a real scalar function and a and γ are two real constants.

In this paper we consider the perturbed modified KdV equation in the form

$$u_t + au^2u_x + \gamma u_{xxx} = \varepsilon u^4u_x \quad (2)$$

where ε is perturbation parameter.

One can see that if ε goes to zero, Eq. (1) transforms to the general simplified modified KdV. Eq. (2) was introduced by Dey et al. [23] and obtained exact solutions of the perturbed modified KdV equation by using some appropriate ansatz. The aim of this paper is to find exact solution of the perturbed modified KdV equation by the functional variable method.

The rest of this Letter is organized as follows. In Section 2, we describe the functional variable method for solving differential equations. Section 3 contains the application of the method to solve the perturbed modified KdV equation. Finally in section 4 some conclusions are presented.

2. The functional variable method

In [24-25], Zerarka et al. introduced the so-called functional variable method to find the exact solutions for a wide class of linear and nonlinear wave equations. This method will play an important role in expressing the traveling wave solutions in terms of hyperbolic, trigonometric and the rational functions for the nonlinear evolution equations in mathematical physics. The advantage of this method is that one treats nonlinear problems by essentially linear methods. The functional variable method has been successfully applied to many kinds of nonlinear differential equations such as, Aminikhah et al., [26] proposed the functional variable method to solve the the

generalized Drinfel'd-Sokolov-Wilson system, Bogoyavlenskii equations and Davey-Stewartson equations. Bekir et al., [27] solved nonlinear time fractional KdV, time fmKdV and time-space fractional Boussinesq equations by using the functional variable method. Nazarzadeh et al., [28] used the functional variable method to obtain the exact solutions of the the generalized forms of Klein–Gordon equation, the (2+1)-dimensional Camassa-Holm Kadomtsev-Petviashvili equation and the higher-order nonlinear Schrödinger equation. Eslami et al., [29] applied the functional variable method to obtain the exact solutions of Davey-Stewartson equation, generalized Zakharov equation, $K(m, n)$ equation with generalized evolution term, (2+1)-dimensional long-wave-short-wave resonance interaction equation and nonlinear Schrödinger equation with power law nonlinearity.

The general characteristics of the functional variable method can be outlined as follows. A nonlinear partial differential equation with several independent variables can be written in the form of

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0 \tag{3}$$

where $u = u(x, t)$ is the solution of nonlinear partial differential equation Eq. (3), the subscript denotes partial derivative and P is a polynomial in its arguments. Zerarka et al, in [24] has summarized the functional variable method in the following.

First of all, the wave transformation can be written as

$$\xi = x - vt \tag{4}$$

where v is constant. Next, we can introduce the following transformation for a travelling wave solution of Eq. (3),

$$u(x, t) = U(\xi) \tag{5}$$

and the chain

$$\frac{\partial}{\partial t}(\cdot) = -v \frac{d}{d\xi}(\cdot) \tag{6-1}$$

$$\frac{\partial}{\partial x}(\cdot) = \frac{d}{d\xi}(\cdot) \tag{6-2}$$

$$\frac{\partial}{\partial t}(\cdot) = \frac{d}{d\xi}(\cdot) \tag{6-3}$$

$$\frac{\partial^2}{\partial x \partial t}(\cdot) = -v \frac{d^2}{d\xi^2}(\cdot) \tag{6-4}$$

Using Eq. (5) and Eq. (6), the nonlinear partial differential equation (3) can be transformed into an ordinary differential equation of the form

$$G(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, \dots) = 0 \tag{7}$$

Then we make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_\xi = F(U) \tag{8}$$

then, the solution can be found by the relation

$$\int \frac{dU}{F(U)} = \xi + \xi_0 \tag{9}$$

here ξ_0 is a constant of integration which is set equal to zero for convenience. Some successive differentiations of U in terms of F are given as

$$U_{\xi\xi} = \frac{1}{2}(F^2)' \tag{10-1}$$

$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2} \tag{10-2}$$

$$U_{\xi\xi\xi\xi} = \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'] \tag{10-3}$$

where $F' = \frac{dF}{dU}$, $F'' = \frac{d^2F}{dU^2}$ and soon.

The ordinary differential equations (5) can be reduced in terms of U , F and its derivatives upon using the expressions of Eq. (10) into Eq. (3) gives

$$G(U, F, F', F'', F''', \dots) = 0 \tag{11}$$

The key idea of this particular form Eq. (11) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq. (11) provides the expression of F and this, together with Eq. (8), give appropriate solutions to the original problem.

3. Exact solutions of the perturbed modified KdV equation

In this Section we obtain traveling wave solutions of the perturbed modified KdV equation by using the functional variable method described in Section 2.

Using the transformation $u(x, t) = U(\xi)$, where ξ is defined in Eq. (4), the Eq. (2) is carried to a ordinary differential equation

$$\gamma U_{\xi\xi} - vU + \frac{\alpha}{3}U^3 - \frac{\epsilon}{5}U^5 = 0 \tag{12}$$

or

$$U_{\xi\xi} = \frac{\nu}{\gamma}U - \frac{\alpha}{3\gamma}U^3 + \frac{\varepsilon}{5\gamma}U^5 \quad (13)$$

Following Eq. (10), it is easy to deduce from Eq. (13) an expression for the function $F(U)$

$$\frac{1}{2}(F^2)' = \frac{\nu}{\gamma}U - \frac{\alpha}{3\gamma}U^3 + \frac{\varepsilon}{5\gamma}U^5 \quad (14)$$

Integrating Eq. (14) and setting the constant of integration to zero yields

$$F = \pm\sqrt{aU^2 + bU^4 + cU^6} \quad (15)$$

where $a = \frac{\nu}{\gamma}$, $b = -\frac{\alpha}{3\gamma}$ and $c = \frac{\varepsilon}{5\gamma}$.

From (15) and (8), we obtain the desired solution as [30-31].

Case 1. If $a > 0$ and $b^2 - 4ac > 0$, then (15) admits the following hyperbolic function solution

$$U(\xi) = \pm \left[\frac{\pm 2a \operatorname{sech}(2\sqrt{a}\xi)}{\sqrt{b^2 - 4ac} \mu b \operatorname{sech}(2\sqrt{a}\xi)} \right]^{\frac{1}{2}} \quad (16)$$

Case 2. If $a > 0$ and $b^2 - 4ac < 0$, then (15) admits the following hyperbolic function solution

$$U(\xi) = \pm \left[\frac{\pm 2a \operatorname{csch}(2\sqrt{a}\xi)}{\sqrt{4ac - b^2} \mu b \operatorname{csch}(2\sqrt{a}\xi)} \right]^{\frac{1}{2}} \quad (17)$$

Case 3. If $a > 0$ and $b^2 - 4ac = 0$, then (15) admits the following hyperbolic function solution

$$U(\xi) = \pm \left[\frac{1}{2} \sqrt{\frac{a}{c}} (1 \pm \tanh(\sqrt{a}\xi)) \right]^{\frac{1}{2}} \quad (18)$$

$$U(\xi) = \pm \left[\frac{1}{2} \sqrt{\frac{a}{c}} (1 \pm \coth(\sqrt{a}\xi)) \right]^{\frac{1}{2}} \quad (19)$$

Case 4. If $a < 0$ and $b^2 - 4ac > 0$, then (15) admits the following trigonometric function solution

$$U(\xi) = \pm \left[\frac{\pm 2a \sec(2\sqrt{-a}\xi)}{\sqrt{b^2 - 4ac} \mu b \sec(2\sqrt{-a}\xi)} \right]^{\frac{1}{2}} \quad (20)$$

$$U(\xi) = \pm \left[\frac{\pm 2a \csc(2\sqrt{-a}\xi)}{\sqrt{4ac - b^2} \mu b \csc(2\sqrt{-a}\xi)} \right]^{\frac{1}{2}} \quad (21)$$

Case 5. If $a > 0$, $b = 2\sigma a$ and $c = (\sigma^2 - 1)a$, then (15) admits the following hyperbolic function solution

$$U(\xi) = \pm \left[\frac{\pm 1}{\cosh(2\sqrt{a}\xi) \mu \sigma} \right]^{\frac{1}{2}} \quad (22)$$

where $\sigma < 1$.

Case 6. If $a > 0$, $b = 2\sigma a$ and $c = (\sigma^2 + 1)a$, then (15) admits the following hyperbolic function solution

$$U(\xi) = \pm \left[\frac{\pm 1}{\sinh(2\sqrt{a}\xi) \mu \sigma} \right]^{\frac{1}{2}} \quad (23)$$

where $\sigma < 1$.

Using the travelling wave transformation (4) and the relations (16-23), we obtain the following travelling wave solutions of the perturbed modified KdV equation (1)

If $\frac{\nu}{\gamma} > 0$ and $5\alpha^2 - 48\nu\varepsilon > 0$, we obtain the

following hyperbolic function solution $u(x, t)$

$$u_{1,2}(x, t) = \pm \left[\frac{\pm 12\nu \operatorname{sech}\left(2\sqrt{\frac{\nu}{\gamma}}(x - vt)\right)}{\frac{1}{5}\sqrt{25\alpha^2 - 240\nu\varepsilon} \mu \alpha \operatorname{sech}\left(2\sqrt{\frac{\nu}{\gamma}}(x - vt)\right)} \right]^{\frac{1}{2}} \quad (24)$$

The solitary wave solution $u_1(x, t)$ in Eq. (24) is shown graphically in Figs. 1.

If $\frac{\nu}{\gamma} > 0$ and $5\alpha^2 - 48\nu\varepsilon < 0$, we obtain the

following hyperbolic function solution $u(x, t)$

$$u_{3,4}(x,t) = \pm \left[\frac{\mu 12v \operatorname{csch}\left(2\sqrt{\frac{v}{\gamma}}(x-vt)\right)}{\frac{1}{5}\sqrt{240v\varepsilon - 25\alpha^2} \pm \alpha \operatorname{csch}\left(2\sqrt{\frac{v}{\gamma}}(x-vt)\right)} \right]^{\frac{1}{2}} \quad (25)$$

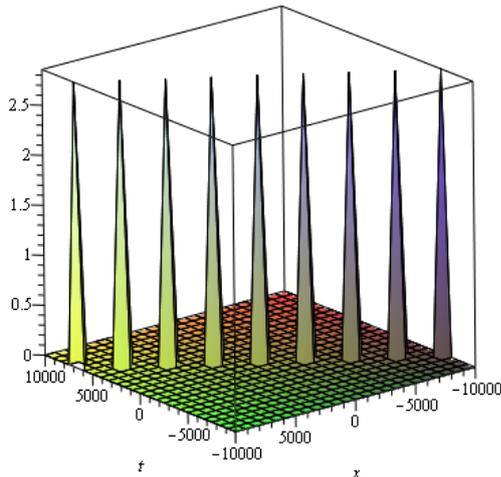


Fig. 1. The exact solution (2) for $a = 6, g = 2, \delta = 1.5$ and $v = 1.5$.

If $\frac{v}{\gamma} > 0$ and $5\alpha^2 - 48v\varepsilon = 0$, we obtain the following hyperbolic function solution $u(x,t)$

$$u_{5,6}(x,t) = \pm \frac{1}{4} \left[\frac{10\alpha(1 \pm \tanh(\frac{1}{12}\sqrt{\frac{15\alpha^2}{\varepsilon\gamma}}(x - \frac{5\alpha^2}{48\varepsilon}t)))}{\varepsilon} \right]^{\frac{1}{2}} \quad (26)$$

$$u_{7,8}(x,t) = \pm \frac{1}{4} \left[\frac{10\alpha(1 \pm \coth(\frac{1}{12}\sqrt{\frac{15\alpha^2}{\varepsilon\gamma}}(x - \frac{5\alpha^2}{48\varepsilon}t)))}{\varepsilon} \right]^{\frac{1}{2}} \quad (27)$$

The solitary wave solution $u_6(x,t)$ in Eq. (26) is shown graphically in Fig. 2.

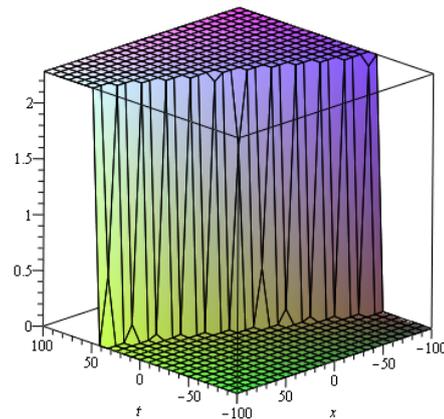


Fig. 2. The exact solution (2) for $a = 5, g = 2$ and $\delta = 1.2$.

If $\frac{v}{\gamma} < 0$ and $5\alpha^2 - 48v\varepsilon > 0$, we obtain the following trigonometric function solution $u(x,t)$

$$u_{9,10}(x,t) = \pm \left[\frac{\mu 12v \sec\left(2\sqrt{-\frac{v}{\gamma}}(x-vt)\right)}{\frac{1}{5}\sqrt{25\alpha^2 - 240v\varepsilon} \pm \alpha \sec\left(2\sqrt{-\frac{v}{\gamma}}(x-vt)\right)} \right]^{\frac{1}{2}} \quad (28)$$

$$u_{11,12}(x,t) = \pm \left[\frac{\mu 12v \csc\left(2\sqrt{-\frac{v}{\gamma}}(x-vt)\right)}{\frac{1}{5}\sqrt{25\alpha^2 - 240v\varepsilon} \pm \alpha \csc\left(2\sqrt{-\frac{v}{\gamma}}(x-vt)\right)} \right]^{\frac{1}{2}} \quad (29)$$

If $\frac{v}{\gamma} > 0$, $v = \frac{\varepsilon}{15(\sigma^2 - 1)}$ and $\alpha = -\frac{4\sigma\varepsilon}{5(\sigma^2 - 1)}$, we obtain the following hyperbolic function solution $u(x,t)$

$$u_{13,14}(x,t) = \pm \left[\frac{\pm 1}{\cosh\left(2\sqrt{\frac{\varepsilon}{15(\sigma^2 - 1)\gamma}}\left(x - \frac{\varepsilon}{15(\sigma^2 - 1)}t\right)\right) \mu \sigma} \right]^{\frac{1}{2}} \quad (30)$$

where $\sigma < 1$.

If $\frac{v}{\gamma} > 0$, $v = \frac{\varepsilon}{15(\sigma^2 + 1)}$ and $\alpha = -\frac{4\sigma\varepsilon}{5(\sigma^2 + 1)}$, we obtain the following hyperbolic function solution $u(x,t)$

$$u_{15,16}(x,t) = \pm \left[\frac{\pm 1}{\sinh\left(2\sqrt{\frac{\varepsilon}{15(\sigma^2+1)\gamma}}\left(x - \frac{\varepsilon}{15(\sigma^2+1)}t\right)\right) \mu \sigma} \right]^{\frac{1}{2}} \quad (31)$$

where $\sigma < 1$.

The solution $u_{15}(x,t)$ in Eq. (31) is represented in Fig. 3.

Remark. The functional variable method definitely can be applied to nonlinear partial differential equations which can be converted to a second-order ordinary differential equation, the travelling wave transformation.

4. Conclusion

In this paper, we have seen that two types of exact analytical solutions including the hyperbolic function solutions and trigonometric function solutions for the perturbed modified KdV equation are successfully found out via the functional variable method. From our results obtained in this paper, we conclude that the functional variable method is powerful, effective and convenient for nonlinear partial differential equations. Also, the solutions of the proposed nonlinear partial differential equations in this paper have many potential applications in physics and engineering. To the best of our knowledge, the solutions obtained in this paper have not been reported in literature.

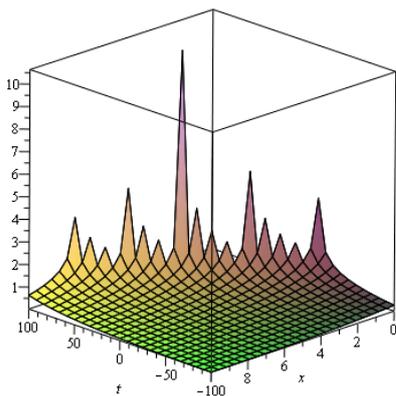


Fig. 3. The exact solution (2) for $a = 6, g = 1, \delta = 1.2$ and $s = 0.5$.

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