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Formation and stability of light bullets: recent theoretical studies

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The spatiotemporal optical solitons (alias "light bullets") are nondiffracting and nondispersing wavepackets propagating in nonlinear optical media. They are localized (self-guided) in two transverse (spatial) dimensions and in the direction of propagation due to the balance of anomalous group-velocity dispersion of the medium in which they form and nonlinear self-phase modulation. The formation of fully three-dimensional light bullets is one of the most exciting, yet experimentally unsolved problems in nonlinear photonics. A brief up-to-date survey of recent theoretical studies of light bullet formation and stability in various physical settings is given.

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1. Introduction

Solitons, or more properly, solitary waves are ubiquitous in nature and have been identified in plasmas, fluids, optics, atomic Bose-Einstein condensates, biologic matter, etc. However, in the past two decades there has been an increasing interest in both theoretical and experimental study of localized light structures (alias, optical solitons), which overcome either dispersion or diffraction [1]-[3]. The temporal or spatial optical solitons are special cases of a larger class of nonlinear confined light structures in which both temporal and spatial effects are coupled and occur simultaneously. The space-time coupling occurring when a pulsed optical beam propagates through a nonlinear optical medium leads to unique nonlinear effects, such as the spatiotemporal collapse for anomalous group-velocity dispersion (GVD) regime, pulse splitting if the GVD of the optical medium is normal, and the formation of confined, in both transverse spatial dimensions, light pulses, which were termed *spatiotemporal optical solitons* [1], [4]-[6].

Typically, these localized multidimensional optical structures are spatially confined on the order of wavelength; however it is possible to get sub-wavelength solitons in metamaterials containing nanostructured noble metal rods embedded in specific nonlinear dielectric media. It is worthy to mention that the study of these plasmon solitons is now emerging as a distinct research direction in the area of nonlinear nano-plasmonics [7]. Optical solitons represent the "particle-like" counterpart of the more common extended light structures. The optical media that might sustain such self-guiding structures should be nonlinear, i.e., their refractive index should depend on the light intensity. Different kinds of nonlinearities of optical materials such as absorptive, dispersive, second-order (quadratic), third-order (Kerr-like) can be used to prevent temporal dispersion/spatial diffraction of light beams or both of them. The field of temporal/spatial optical solitons emerged from these fundamental studies of interaction of intense laser beams with matter.

The research area of optical solitons is now in a mature stage; temporal optical solitons are currently created in monomode optical fibers and have led to a mature photonics technology, whereas spatial optical solitons are currently created in laboratory and are now awaiting technological implementation in all-optical information processing. However, the spatiotemporal optical solitons, alias “light bullets” (LBs), constitute the third kind of optical solitons [4,5]. They are spatially confined pulses of light, i.e., electromagnetic wave packets self-trapped in both space and time and could be used as information carriers in future all-optical processing information systems. It is worthy to mention that the optical solitons in media with a cubic self-focusing nonlinearity, obeying the nonlinear Schroedinger (NLS) equation, are unstable in two and three dimensions, because of the occurrence of optical beam collapse [8,9]. However, several possibilities to arrest the intrinsic wave collapse were considered, such as the use of quadratic nonlinear optical media that support solitons for all physical dimensions [10]-[13]. The (2+1)-dimensional light bullet formation was achieved in quadratic nonlinear crystals by generating the necessary anomalous GVD via achromatic phase matching [14]. Other physical settings, which are adequate for getting stable light bullet formation use saturable [15,16] and nonlocal [17,18] optical media, materials with competing nonlinearities [19,20], confining two- or three-dimensional optical lattices [21]-[25], and periodic (discrete) waveguide structures with controlled diffraction and/or GVD [26]-[29]. The formation of multidimensional fundamental and vortex (spinning) dissipative solitons in media with gain and loss described by the cubic-quintic Ginzburg-Landau equation [30,31], and the existence of stable *discrete light bullets* in one- and two-dimensional photonic lattices [32]-[34] were also put forward .

The landmark experimental work [14] reporting the formation of a (2+1)-dimensional spatiotemporal optical soliton used a very clever scheme to control the GVD along one spatial axis. The beam self-trapping occurred only along one spatial transverse dimension of a two-dimensional optical beam. It is well known that by reflecting a beam from a diffraction grating, the nonspecular orders have their energy wavefront tilted relative to their phase velocity wavefront, with different spectral components having different tilts; pulse compression in time based on this principle was achieved by using the cascaded nonlinearity in second-order nonlinear optical materials, such as lithium iodate and beta-barium borate (BBO) [14]. Quadratic spatiotemporal solitons in the cascaded limit with highly elliptically shaped beams were generated by using the above mentioned second-harmonic generation crystals. Along the long axis of the optical beam cross-section, the diffraction length was longer than the length of the crystal so that no beam diffraction occurred. However, along the short beam axis, the diffraction length was about one fifth of the crystal length and it is along this transverse coordinate that the beam behaved like a spatial optical soliton. The pulse width of about 100 fs was used in this experiment, with the grating-engineered GVD, to match the dispersion length to the diffraction length in order to form a spatiotemporal optical soliton ("light bullet"). It was demonstrated that along the short beam axis no spreading occurred both in space and in time, a characteristic feature of a (2+1)-dimensional light bullet. Thus for propagation over five characteristics lengths, the beam size (pulse duration) was about 50 microns (100 fs). It is worthy to mention that in this experiment it was also reported, for the first time to my knowledge, the formation and the propagation over several dispersion lengths of temporal solitons in quadratic nonlinear optical media [14].

This work is organized as follows. In Sec. 2 I briefly overview the studies of existence, stability and robustness of three-dimensional vortex solitons (vortex tori) in both conservative and dissipative settings. The problem of formation of stable three-dimensional light bullets in lower dimensional photonic lattices is discussed in Sec. 3. Section 4 is devoted to the discussion of a few recent, innovative physical settings, which are able to support the formation of stable spatiotemporal optical solitons. Finally, Sec. 5 concludes the paper.

2. Stable spatiotemporal spinning solitons

One peculiar feature of wavefields is the occurrence of vortices within them. A vortex is a singular point in the wavefield around which there is a continuous circulation of a certain physical quantity. In optics, the localized optical vortices (alias *optical vortex solitons*), have drawn much attention as objects of fundamental interest, and also due to their potential applications to all-optical information processing, as well as to the guiding and trapping of atoms. In the core of an optical vortex the complex electromagnetic field is equal to zero, however the circulation C of the gradient of the phase of the complex field on an arbitrary closed contour around the vortex core is a multiple of 2π, i.e., $C=2\pi S$, where the integer S is the *topological number* of the vortex ("spin"). Thus the phase dislocations carried by the wavefront of a light beam are associated with a zero-intensity point (a vortex core); the phase is twisted around such points where the light intensity vanishes, creating an optical vortex. It is worthy to mention that unique properties are also featured by vortex clusters, such as rotation similar to the vortex motion in superfluids. The complex dynamics of two- and three-dimensional soliton clusters in optical media with competing nonlinearities has been studied too [35]-[37]. Various complex patterns based on both fundamental (nonspinning) solitons and vortices were theoretically investigated in optics and in the usual BEC models governed by the Gross-Pitaevskii equation with both local [38]-[40] and nonlocal nonlinearity [41].

Stable nondissipative spatiotemporal spinning solitons (vortex tori) with the topological charge S=1 (see Fig. 1), described by the three-dimensional NLS equation with focusing cubic and defocusing quintic nonlinearities were found to exist for sufficiently large energies [19]. This result also holds for the case of competing quadratic and self-defocusing cubic nonlinearities [20]. A general conclusion of these studies is that stable spinning solitons are possible as a result of competition between focusing and defocusing optical nonlinearities.

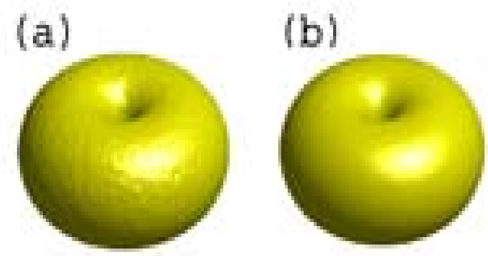

Fig. 1. Recovery of a light bullet with "spin" S=1 propagating in a cubic-quintic optical medium, which was perturbed at input (10% random noise): (a) isosurface plot of the initially perturbed vortex torus and (b) the self-cleaned output soliton. Here the nonlinear wave number k=0.15 (see Ref. 19).

We have also performed a comprehensive stability analysis of three-dimensional dissipative solitons with intrinsic vorticity S governed by the complex Ginzburg-Landau equation with cubic and quintic terms in its dissipative and conservative parts [30,31]. It was found that a necessary stability condition for all vortex solitons, but not for the fundamental ones (S=0), is the presence of nonzero diffusivity in the transverse plane. The fundamental solitons are stable in all cases when they exist, while the vortex solitons are stable only in a part of their existence domain. However, the spectral filtering (i.e., the temporal-domain diffusivity) is not necessary for the stability of any species of dissipative solitons. Stability domains were found for (3+1)-dimensional vortex solitons (vortex tori) with "spin" S=1, 2, and 3, suggesting that spinning solitons with any vorticity S can be stable in certain portions of their existence domains [31]. It is worthy to mention that the signature of an optical vortex with topological charge S can be detected by looking at the unique structure of the interference pattern of the vortex field with a plane wave (these interferograms display typical "fork-like" dislocations in the vortex core). The stable vortex torus is a robust physical object, i.e., it is able to absorb the white noise perturbation and to clean up itself. The spinning (vortex) soliton can be easily generated from a Gaussian spatiotemporal input electromagnetic field with a nested vortex; the Gaussian optical field with a phase dislocation at the vortex core evolves towards a stable flat-top like vortex torus, see Ref. [30,31]. It is worthy to mention that in fluid mechanics it was experimentally observed [42,43] the formation of toroidal viscoelastic drops, resembling optical vortex tori. Thus a stable toroidal drop of viscoelastic fluid sinking through a viscous Newtonian oil was produced. In such a highly dissipative medium with a very small Reynolds number, a Newtonian vortex ring would be unstable (growing in size and slowing down), so it is the additional elasticity in the fluid drop that provided a sort of a stabilizing effect [42,43].

The study of localized structures in optics has recently identified *nonlinear self-similar propagation* as a robust means of avoiding optical beam or pulse breakup at high power [44]-[47]. Spatiotemporally expanding self-similar light bullets and vortex torus solutions to the three-dimensional NLS equation with gain were introduced recently [48]. In the absence of an initial vorticity, it was demonstrated an *expanding similariton* with a parabolic intensity profile and linear spatiotemporal chirp. Expanding vortex torus solutions with a centrally embedded phase singularity were also found. Furthermore, it was shown by extensive numerical simulations that these self-similar solutions of NLE equation with gain are nonlinear attractors towards which arbitrarily shaped input pulses converge asymptotically [48].

3. Stable spatiotemporal solitons in two-dimensional photonic lattices

A very promising way to arrest the collapse in cubic (Kerr-type) focusing media is to use two-dimensional nonlinear photonic lattices in a three-dimensional environment [21]-[25]. The existence and stability of three-dimensional spatiotemporal solitons in self-focusing cubic Kerr-type optical media with an imprinted two-dimensional harmonic transverse modulation of the refractive index was studied in detail in Ref. [22]. It was demonstrated that two-dimensional photonic Kerr-type nonlinear lattices can support stable one-parameter families of three-dimensional spatiotemporal solitons provided that their energy is within a certain interval and the strength p of the lattice potential, which is proportional to the refractive index modulation depth, is above a certain threshold value. As a consequence of the imprinted two-dimensional photonic lattice, the nonlinear localized states exist only for nonlinear wave numbers (propagation constants) larger than some minimum values (the edge of the band gap). The minimum propagation constant increases with the increase of the lattice strength parameter; recall that for the NLS equation the minimum propagation constant is equal to zero. Families of three-dimensional spatiotemporal solitons in two-dimensional harmonic lattices exist whenever their energy exceeds a certain minimum value and are linearly stable in the intermediate-energy regime and for sufficiently high lattice strengths. Remarkably, for sufficiently large values of the lattice strength parameter p, the Hamiltonian-versus-energy (soliton norm) curves display two cusps (see Fig. 2), instead of a single one as in other 2D and 3D nondissipative (Hamiltonian) nonlinear dynamical systems. This unique two-cusp structure of the soliton norm-Hamiltonian diagram is the so-called *"swallowtail"* catastrophe and is quite rare in physics [23,24].

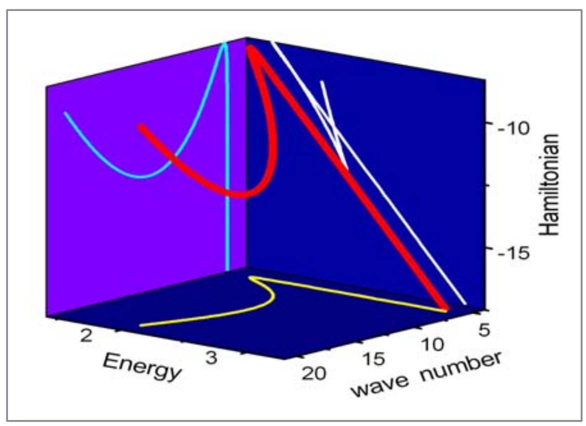

Fig. 2. Typical energy (soliton norm)-wave number-Hamiltonian diagram for 3D light bullets confined by 2D optical lattices. Here the lattice strength parameter is p=20 (see Ref. 22).

Remarkably, this unique swallowtail bifurcation occurs also in the study of stability of three-dimensional solitons with vorticity S=1 supported by a two-dimensional harmonic lattice if the lattice strength is large enough [25]. Recently we have introduced *discrete surface light bullets* forming in both one-dimensional [32] and two-dimensional [33] photonic lattices. We analyzed spatiotemporal light localization near the edge of semi-infinite arrays of weakly coupled nonlinear optical waveguides or in the corners and the edges of two-dimensional photonic lattices and demonstrated the existence and stability (in certain regions of their existence domain) of continuous-discrete spatiotemporal surface solitons [32,33]. We have shown that their properties, such as power (energy) thresholds for their formation are strongly affected by the presence of the photonic lattice truncation. Recently we analyzed the interactions between discrete surface light bullets and we observed a variety of collision scenarios and different outcomes, such as soliton fusion, soliton switching, symmetric and asymmetric scattering [34].

4. Recent developments

Trains of spatiotemporal localized structures have been recently predicted in Raman-active optical media due to spatiotemporal coupling induced by the four-wave mixing phenomenon [49]. It is worthy to mention that in the majorities of studies of light bullet formation the same nonlinear mechanism counteracts beam difraction and GVD, implying that he characteristic length of difraction, dispersion and nonlinearity must be matched as a necessary condition for the formation of spatiotemporal solitons. However, recently it was put forward new approach for constructing spatiotemporal pulse-train solitons, which is based on employing slow nonlinearities for removing the condition that the dispersion length must be equal to the diffraction length [50]. The pulses were collectively trapped in the transverse direction by a slow nonlinearity, which facilitates the soliton stability, and each pulse is self-trapped in the longitudinal direction by a fast nonlinearity. In Ref. [50] the characteristic length of the slow nonlinearity corresponds to the diffraction length, while the length of the fast nonlinearity matches the dispersion length (the diffraction length is much shorter than the dispersion length in this physical setting).

Another possibility to get stable light bullets comes from the interplay of local and nonlocal nonlinearities [51]. The concept of accessible light bullets via synergetic optical nonlinearities (nonlocal, non-instantaneous nonlinearity combined with an instantaneous Kerr-type nonlinear response) was put forward in Ref. [51]. By extensive numerical calculations it was demonstrated that (3+1)-dimensional light bullets and anti-bullets can be generated in reorientational media with a cubic electronic nonlinearity, such as liquid crystals in the nematic phase, under experimentally feasible conditions [51].

A potential approach to form stable 3D light bullets might be based on the concept of engineered structures composed of different optical materials featuring either strong nonlinearity or strong suitable GVD but not necessarily both together at a given wavelength [52]. The implementation of such idea along the propagation (longitudinal) direction showed that light bullet formation is possible for significantly large tandem domains in the case of quadratic spatiotemporal solitons [52].

Very recently, it was shown that stable 3D light bullets do form in transverse radially periodic metamaterial structures consisting of alternating rings made of highly dispersive linear materials and rings made of strongly nonlinear media (with cubic saturable optical nonlinearities) [53]. It was found that light bullet stability depends crucially on whether the central domain is linear or nonlinear; though stabilization in optical tandems with a nonlinear central domain is still possible, the corresponding stability region in terms of nonlinear wave number is quite narrow, and stabilization occurs at much higher powers than in optical tandems with a linear central domain. The key result reported in Ref. [53] is that (3+1)-dimensional light bullets do form in suitable metamaterial structures where the different materials are used at their best to meet the requirements needed to get light bullets in practice.

As concerning the possible practical implementation of the light bullet concept we mention here a quite realistic physical setting involving silicon nanowires [54]. The conditions for low-power spatiotemporal soliton formation in arrays of evanescently-coupled silicon-on-insulator (SOI) photonic nanowires have been thoroughly analyzed recently [54]. It was shown that pronounced soliton effects can be observed even in the presence of realistic loss, two-photon absorption, and higher-order GVD. The well established SOI technology offers an exciting opportunity in the area of spatiotemporal optical solitons because a strong anomalous GVD can be achieved with nanoscaled transverse dimensions and moreover, the enhanced nonlinear response resulting from this tight transverse spatial confinement of the electromagnetic field leads to soliton peak powers of only a few watts for 100-fs pulse widths (the corresponding energy being only a few hundreds fJ). It is worthy to note that the arrays of SOI photonic nanowires seem to be most adequate for the observation of discrete surface light bullets because a suitable design of nanowaveguides can provide dispersion lengths in the range of 1 mm and coupling lengths of a few millimeters (for 100-fs pulse durations) [54].

The study of fully three-dimensional light bullets in materials with simultaneous modulation of the linear refractive index and of the nonlinearity is of increasing interest [55] because the relevant physical settings where stable three-dimensional solitons exist are relatively rare [4]. Moreover, since three-dimensional solitons in cubic (Kerr-like) nonlinear media suffer from supercritical collapse [8]-[9], addition of a linear optical lattice results in their stabilization only in a certain limited range of relevant parameters. It is therefore expected that the additional modulation of the cubic optical nonlinearity may dramatically affect the domains of existence and stability of spatiotemporal optical solitons. Recently, it was investigated an interesting physical setting with competing linear and nonlinear lattices, where the refractive index is decreased in the points where the nonlinearity is strongest [55]. Thus it was investigated the stability of spatiotemporal optical solitons supported by Bessel optical lattices with out-of-phase modulation of the linear and nonlinear refractive indices. It was shown that the spatial modulation of the nonlinear refractive index significantly modifies both the shapes and the stability domains of the light bullets. However, it has been proven that the fully three-dimensional light bullets forming in a two-dimensional Bessel lattice can be stable, provided that the peak intensity does not exceed a certain critical value [55]. Moreover, it was found that the width of the stability domain in terms of the propagation constant may be controlled by varying the nonlinearity modulation depth and that the maximum energy at which the spatiotemporal optical solitons remain stable increases with the depth of the nonlinearity modulation. An interesting result obtained in Ref. [55] is that the so-called Vakhitov-Kolokolov stability criterion accurately predicts the domains of stability and instability only when the nonlinearity modulation depth parameter is not too strong.

As I said before, spatiotemporal solitons forming in dissipative media (*dissipative light bullets*) have also attracted a lot of attention in the past few years. It is worthy to mention a recent work where spatiotemporal necklace-shaped patterns with annular or radial phase modulation were theoretically studied [56] in two relevant physical settings: (a) dissipative media governed by the three-dimensional complex Ginzburg-Landau equation, and (b) dissipative systems described by the three-dimensional complex Swift-Hohenberg equation. It was demonstrated that spatiotemporal necklace-shaped patterns with annular phase modulation can fuse into stable fundamental or vortex solitons in both dynamical dissipative models mentioned above, when the initial radius of the necklace is smaller than a critical value, which is similar to the fusion of two-dimensional necklace-shaped patterns into stable fundamental or vortex solitons in a similar complex Ginzburg-Landau equation, see, e.g., Ref. [57]. Moreover, it was shown in Ref. [56] that when a radial phase modulation is added to the spatiotemporal necklace-shaped pattern, the modulated "bead" (that is, one of the elements forming the ring structure) will move either towards or off the center of the necklace and will rapidly vanish due to the presence of dissipation in the system. The formation of stable dissipative light bullets in the degenerate optical parametric oscillator was also studied [58]. We recently considered continuous-discrete spatiotemporal models described by the complex Ginzburg-Landau equation [59]-[60]. Thus the presence of gain and loss due to optical amplifiers and saturable absorbers in truncated one- and two-dimensional periodic photonic structures has been investigated and dissipative surface LBs were introduced in both one-dimensional waveguide arrays [59] and in two-dimensional photonic lattices [60]. The domains of existence and stability of in-phase (unstaggered) on-site (single-peaked), inter-site (double-peaked) and flat-top-like (four-peaked) dissipative LBs in 2D photonic lattices were determined and the various instability-induced scenarios of the dynamics of these discrete Ginzburg-Landau spatiotemporal optical solitons were described [61]. Further, systematic results of collisions between discrete spatiotemporal Ginzburg-Landau solitons were reported [62]. The generic outcomes of collisions between both co-rotating [63] and counter-rotating [64] vortex solitons and between nonspinning and spinning [65] co-axial 3D dissipative LBs described by the complex Ginzburg-Landau equation with the cubic-quintic nonlinearity were recently presented.

It is also relevant to mention the study of the so-called nonlinear X-waves [66]. This is a generalization of the diffraction-free propagation of Bessel beams [67] in a linear medium to the polychromatic case. Thus it was discovered [66] that a quasilocalized light pattern can spontaneously emerge from unstable propagation of a short pulse in a quadratic nonlinear crystal with normal GVD and large group-and phase-velocity mismatches. Recently, direct spatiotemporal measurements of accelerating and decelerating ultrashort Bessel-type LBs with micron spatial resolution and femtosecond temporal resolution have been performed [68]. The existence of generic LBs compressed to the few-cycle limit in the filamentation regime with no external compressor system has been numerically demonstrated [69]. These few-cycle LBs can be formed in gaseous as well as dense media. Thus by coupling an infrared pump with a seed beam, tunable pulses with durations down to a few femtoseconds can be generated by parametric processes and propagate over long distances with a stable profile [69]. The observation of 3D discrete-continuous X-waves in photonic lattices (femtosecond laser-written waveguide arrays) has been reported recently [70]. These measurements constitute, to the best of my knowledge, the first experimental observation of temporally localized 3D discrete-continuous entities. Comprehensive theoretical studies of the existence, stability and interactions of spatiotemporal solitons and vortices in optical fiber bundles have been also reported [71]. In connection with these theoretical studies, the dynamics of spatiotemporal nonlinear localization in arrays of evanescently coupled silica fiber arrays, which form two-dimensional waveguide lattices, has been recently investigated and it was shown that in contrast to continuous systems the formation of stable LBs becomes possible [72].

5. Conclusions

I conclude with the hope that this brief overview on recent exciting theoretical developments in the area of multidimensional localized structures in optics will inspire further theoretical and experimental investigations. Also, I do believe that interesting times have arrived for the field of light bullets.

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References

[1] Y. S. Kivshar, G. P. Agrawal, Academic Press, San Diego, 2003.
[2] N. Akhmediev, A. Ankiewicz (Eds.), Lect. Notes Phys. vol. 751, Springer, Berlin, 2008.
[3] G. I. Stegeman, D. N. Christodoulides, M. Segev, IEEE J. Select. Top. Quant. Electron. **6**, 1419 (2000).
[4] B. A. Malomed, D. Mihalache, F. Wise, L. Torner, J. Opt. B: Quantum Semiclass. Opt. **7**, R53 (2005).
[5] Y. Silberberg, Opt. Lett. **15**, 1282 (1990).
[6] D. Mihalache, D. Mazilu, Rom. Rep. Phys. **60**, 749 (2008); **60**, 957 (2008); **61**, 175 (2009); **61**, 235 (2009); **61**, 587 (2009).
[7] E. Feigenbaum and M. Orenstein, Opt. Lett. **32**, 674 (2007).
[8] L. Berge, Phys. Rep. **303**, 260 (1998).
[9] L.-C. Crasovan, J. P. Torres, D. Mihalache, L. Torner, Phys. Rev. Lett. **91**, 063904 (2003).
[10] N.-C. Panoiu, R.M. Osgood, B.A. Malomed, F. Lederer, D. Mazilu, and D. Mihalache, Phys. Rev. E **71**, 036615 (2005).
[11] D. Mihalache, D. Mazilu, B.A. Malomed, L. Torner, Opt. Commun. **152**, 365 (1998).
[12] D. Mihalache, F. Lederer, D. Mazilu, L.-C. Crasovan, Opt. Eng. **35**, 1616 (1996).
[13] D. Mihalache, D. Mazilu, L.-C. Crasovan, L. Torner, B. A. Malomed, F. Lederer, Phys. Rev. E **62**, 7340 (2000).
[14] X. Liu, L.J. Qian, F.W. Wise, Phys. Rev. Lett. **82**, 4631 (1999).
[15] D. E. Edmundson, R. H. Enns, Opt. Lett. **17**, 586 (1992).
[16] N. Akhmediev, J. M. Soto-Crespo, Phys. Rev. A **47**, 1358 (1993).
[17] O. Bang, W. Krolikowski, J. Wyller, J. J. Rasmussen, Phys. Rev. E **66**, 046619 (2002).
[18] D. Mihalache, D. Mazilu, F. Lederer, B. A. Malomed, Y. V. Kartashov, L.-C. Crasovan, and L. Torner, Phys. Rev. E **73**, 025601 (2006).
[19] D. Mihalache, D. Mazilu, L.-C. Crasovan, I. Towers, A. V. Buryak, B. A. Malomed, L. Torner, J. P. Torres, F. Lederer, Phys. Rev. Lett. **88**, 073902 (2002).
[20] D. Mihalache, D. Mazilu, L.-C. Crasovan, I. Towers, B. A. Malomed, A. V. Buryak, L. Torner, F. Lederer, Phys. Rev. E **66**, 016613 (2002).
[21] B. B. Baizakov, B.A. Malomed, M. Salerno, Phys. Rev. A **70**, 053613 (2004).
[22] D. Mihalache, D. Mazilu, F. Lederer, Y. V. Kartashov, L.-C. Crasovan, L. Torner, Phys. Rev. E **70**, 055603 (2004).
[23] D. Mihalache, D. Mazilu, F. Lederer, B. A. Malomed, Y. V. Kartashov, L.-C. Crasovan, L. Torner, Phys. Rev. Lett. **95**, 023902 (2005).
[24] D. Mihalache, D. Mazilu, F. Lederer, B. A. Malomed, L.-C. Crasovan, Y. V. Kartashov, L. Torner, Phys. Rev. A **72**, 021601 (2005).
[25] H. Leblond, B.A. Malomed, D. Mihalache, Phys. Rev. E **76**, 026604 (2007).
[26] A. B. Aceves, C. De Angelis, A. M. Rubenchik, S. K. Turitsyn, Opt. Lett. **19**, 329 (1994).
[27] E. W. Laedke, K. H. Spatschek, S. K. Turitsyn, Phys. Rev. Lett. **73**, 1055 (1994).
[28] Z. Xu, Y. V. Kartashov, L. C. Crasovan, D. Mihalache, L. Torner, Phys. Rev. E **70**, 066618 (2004).
[29] A. A. Sukhorukov, Y. S. Kivshar, Phys. Rev. Lett. **97**, 233901 (2006).
[30] D. Mihalache, D. Mazilu, F. Lederer, Y. V. Kartashov, L.-C. Crasovan, L. Torner, B. A. Malomed, Phys. Rev. Lett. **97**, 073904 (2006).
[31] D. Mihalache, D. Mazilu, F. Lederer, H. Leblond, B. A. Malomed, Phys. Rev. A **75**, 033811 (2007); Phys. Rev. A **76**, 045803 (2007).
[32] D. Mihalache, D. Mazilu, F. Lederer, Y.S. Kivshar, Opt. Express **15**, 589 (2007).
[33] D. Mihalache, D. Mazilu, F. Lederer, Y. S. Kivshar, Opt. Lett. **32**, 3173 (2007).
[34] D. Mihalache, D. Mazilu, F. Lederer, Y. S. Kivshar, Phys. Rev. A **79**, 013811 (2009); Opt. Commun. **282**, 3000 (2009).
[35] Y.V. Kartashov, L.-C. Crasovan, D. Mihalache, L. Torner, Phys. Rev. Lett. **89**, 273902 (2002).
[36] L.-C. Crasovan, Y.V. Kartashov, D. Mihalache, L. Torner, Y.S. Kivshar, V.M. Perez-Garcia, Phys. Rev. E **67**, 046610 (2003).
[37] D. Mihalache, D. Mazilu, L.-C. Crasovan, B.A. Malomed, F. Lederer, and L. Torner, J. Opt. B: Quantum Semiclass. Opt. **6**, S333 (2004).
[38] L.-C. Crasovan, G. Molina-Terriza, J.P. Torres, L. Torner, V.M. Perez-Garcia, and D. Mihalache, Phys. Rev. E **66**, 036612 (2002).
[39] L.-C. Crasovan, V. Vekslerchik, V.M. Perez-Garcia, J. P. Torres, D. Mihalache, L. Torner, Phys. Rev. A **68**, 063609 (2003).
[40] D. Mihalache, D. Mazilu, B.A. Malomed, F. Lederer, Phys. Rev. A **73**, 043615 (2006).
[41] Y.J. He, B.A. Malomed, D. Mihalache, H.Z. Wang, Phys. Rev. A **77**, 043826 (2008).
[42] M.C. Sostarecz and A. Belmonte, J. Fluid Mech. **497**, 235 (2003).
[43] M.C. Sostarecz and A. Belmonte, Phys. Fluids **15**, S5 (2003).
[44] F.O. Ilday, J. R. Buckley, W.G. Clark, F. W. Wise, Phys. Rev. Lett. **92**, 213902 (2004).
[45] S.A. Ponomarenko, G. P. Agrawal, Phys. Rev. Lett. **97**, 013901 (2006).
[46] J. M. Dudley, C. Finot, D.J. Richardson, G. Millot, Nature Phys. **3**, 597 (2007).
[47] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, V.V. Konotop, Phys. Rev. Lett. **100**, 164102 (2008).
[48] S. Chen, J. M. Dudley, Phys. Rev. Lett. **102**, 233903 (2009).

[49] A. V. Gorbach, D.V. Skryabin, Phys. Rev. Lett. **98**, 243601 (2007).
[50] H.C. Gurgov and O. Cohen, Opt. Express **17**, 7052 (2009).
[51] I. B. Burgess, M. Peccianti, G. Assanto, R. Morandotti, Phys. Rev. Lett. **102**, 203903 (2009).
[52] L. Torner, S. Carrasco, J.P. Torres, L.-C. Crasovan, D. Mihalache, Opt. Commun. **199**, 277 (2001).
[53] L. Torner, Y. V. Kartashov, Opt. Lett. **34**, 1129 (2009).
[54] C. J. Benton, A.V. Gorbach, D.V. Skryabin, Phys. Rev. A **78**, 033818 (2008).
[55] F. Ye, Y.V. Kartashov, B. Hu, L. Torner, Opt. Express **17**, 11328 (2009).
[56] Bin Liu, Ying-Ji He, Zhi-Ren Qiu, and He-Zhou Wang, Opt. Express **17**, 12203 (2009).
[57] Y. J. He, H. Z. Wang, B. A. Malomed, Opt. Express **15**, 17502 (2007).
[58] N. Veretenov and M. Tlidi, Phys. Rev. A **80**, 023822 (2009).
[59] D. Mihalache, D. Mazilu, F. Lederer, Y. S. Kivshar, Phys. Rev. A **77**, 043828 (2008).
[60] D. Mihalache, D. Mazilu, F. Lederer, Y. S. Kivshar, Phys. Rev. E **78**, 056602 (2008).
[61] D. Mihalache, D. Mazilu, and F. Lederer, Eur. Phys. J. Special Topics **173**, 255 (2009).
[62] D. Mihalache, D. Mazilu, and F. Lederer, Eur. Phys. J. Special Topics **173**, 267 (2009).
[63] D. Mihalache, D. Mazilu, F. Lederer, H. Leblond, and B. A. Malomed, Phys. Rev. A **77**, 033817 (2008).
[64] D. Mihalache, D. Mazilu, F. Lederer, H. Leblond, and B. A. Malomed, Phys. Rev. E **78**, 056601 (2008).
[65] D. Mihalache, D. Mazilu, F. Lederer, H. Leblond, and B. A. Malomed, Eur. Phys. J. Special Topics **173**, 245 (2009).
[66] P. Di Trapani, G. Valiulis, A. Piskarskas, O. Jedrkiewicz, J. Trull, C. Conti, S. Trillo, Phys. Rev. Lett. **91**, 093904 (2003).
[67] P. Saari, K. Reivelt, Phys. Rev. Lett. **79**, 4135 (1997).
[68] H. Valtna-Lukner, P. Bowlan, M. Lohmus, P. Piksarv, R. Trebino, P. Saari, Opt. Express **17**, 14948 (2009).
[69] L. Berge, S. Skupin, Phys. Rev. Lett. **100**, 113902 (2008).
[70] M. Heinrich, A. Szameit, F. Dreisow, R. Keil, S. Minardi, T. Pertsch, S. Nolte, A. Tunnermann, F. Lederer, Phys. Rev. Lett. **103**, 113903 (2009).
[71] H. Leblond, B.A. Malomed, and D. Mihalache, Phys. Rev. A **77**, 063804 (2008); Phys. Rev. A **79**, 033841 (2009).
[72] T. Pertsch et al., 2009 IEEE/LEOS Winter Topicals Meeting Series (WTM 2009), pp. 162-163.

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