Minimum energy principle for electric and magnetic circuits in quasi-stationary regime

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The use of power and energy functionals in the Hilbert space makes it possible to appreciate the energy equilibrium state attained in the quasi-stationary regime of electromagnetic fields at a certain moment. This paper deals with a demonstration of the principle of minimum energy for the electric and magnetic circuits in quasi-stationary regime. It is shown that the equilibrium state of a circuit represent a minimum energy state.

1. Introduction

Conservative systems accept the definition of functionals expressed in terms of power or energy. Calculating the limits of these functionals represents an important breakthrough in formulating and solving optimization problems. The steady-states in mechanic, thermal, and electric conservative systems generally represent limit states from an energy point of view [1]. For example, in classical mechanics, the Hamilton's principle, which refers to the state of minimal action in the time-related evolution of the system from one quasi-stationary state to another, along the curve $\gamma : [a,b] \rightarrow R^n$, is defined by the minimum of the functional

$$F = \int_a^b (T - U) dt \qquad (1)$$

also called the "action integral", where F is the energy functional, T is the kinetic energy of the system, and U is the mechanic effect of the power system which works on the system under consideration. In the theory of the electric circuits, the results obtained by Miller [2] and Stern [3] in connection with the "co-content" function for the non-linear resistive and reciprocal circuits, have a noticeable theoretical importance. Others valuable contributions have appeared in the literature with the works of Desoer and Kuh ([4], pp. 770-772), and Smith [5,6], each of which has shed more light on some aspects of the minimum dissipated power in the linear and resistive circuits. Furthermore, Ionescu [7] and Mocanu ([8] pp. 350-353) provided important contributions to the theoretical development of the electrical circuits minimax theorems. These results are basically consequences of Maxwell's principles of minimum-heat ([9], pp. 407-408). More recently, some papers have addressed the application

of the Hamiltonian formulation and have dealt with the definition of a storage function related to the circuit power [10,11]. Starting from these principles and from the analogies that exist between the magnetic and electric circuits, the present paper aims at demonstrating the minimum energy principle in the electric circuits with capacitors and in the linear magnetic circuits in quasi-stationary regime.

2. The minimum energy principle for electric and magnetic circuits

2.1. DC and AC circuits

Research works published by authors such as [12-16] have thoroughly demonstrated the minimum energy principle for the electric circuits in stationary and quasi-stationary regime.

For DC circuits, the following power functional has been defined in the Hilbert space R^n

$$F : R^n \rightarrow R \ , \quad F \overset{\Delta}{=} \frac{1}{2} [U]^T [I] \qquad (2)$$

where the superscript T denotes transposition, the voltage matrix $[U]$ and the current matrix $[I]$ verify the first and the second theorem of Kirchhoff, and the functional is positive defined. It has been demonstrated that the extreme points of the functional are minimum, and verify the first theorem of Kirchhoff. Consequently, it has been established the following principle:
    "*The minimum of the absorbed power by the branches of a linear and resistive circuit in a stationary regime is verified by the solutions of the currents and voltages in the*

circuit, and these are the currents and voltages that verify the first and the second Kirchhoff's theorems".

The same result can be expressed in the following form:

"In a reciprocal and resistive DC circuit the currents and voltages get distributed such as the power absorbed by the branches of the circuit is minimal".

A similar principle has been demonstrated for the quasi-stationary regime (AC) of a linear electric circuit, yielding the second principle of the minimum active and reactive absorbed power:

"The minimum of active and reactive absorbed (generated) power by the branches of a linear circuit in quasi-stationary AC regime is verified by the solutions in currents and voltages in the circuit, and these are the currents and voltages that verify the first and second Kirchhoff's theorems".

The same principle can be expressed in the following alternative form:

"In the linear and reciprocal circuit in quasi-stationary AC regime the currents and voltages are distributed such as the active and reactive absorbed (generated) power by the branches of the circuit is minimal".

2.2. Circuits with capacitors

We shall now carry out a demonstration of the minimum energy principle in a circuit composed of capacitors. For a given branch of the network connected between two nodes which have the potentials V_x and V_y, made of a capacitor C charged with electric charge q from the voltage source E, as in Fig. 1, we start from the expression of the voltage between the two nodes

$$U = V_x - V_y = U_C + U_E = \frac{q}{C} - E \qquad (3)$$

The electrostatic energy of the system expressed in terms of potentials is

$$W_e = \frac{1}{2}q(V_x - V_y + E) = \frac{1}{2}C(V_x - V_y + E)^2 \quad (4)$$

The electrostatic energy, having values obtained by computing the integral of a vector function, is an energy functional F_E defined within the vector space of the electrical field and having values in the positive real set R_+ ($F_E : S_E \rightarrow R_+$)

$$F_E \equiv W_E = \frac{1}{2}C(V_x - V_y + E)^2 \qquad (5)$$
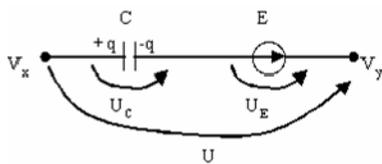
Fig. 1. Branch with capacitor.

The functional is positive defined, i.e., $F_E(V_x, V_y) \geq 0$, whatever the values of the potentials V_x and V_y might be. This means that its extreme points will be minimum points and, therefore, the solutions of the system with partial derivatives

$$\frac{\partial F_E}{\partial V_x} = 0 \qquad (6)$$

and

$$\frac{\partial F_E}{\partial V_y} = 0 \qquad (7)$$

correspond to the minimum electrostatic energy of the network. By computing the system (6) for all the l_k branches connected to node x, we obtain a relation that verifies Kirchhoff's first theorem for circuit with capacitors, expressed at node x:

$$\sum_{l_k \in x} \frac{\partial F_{E_k}}{\partial V_{x_k}} = \sum_{l_k \in x} C_k(V_{x_k} - V_{y_k} + E_k) = \sum_{l_k \in x} q_k = 0 \quad (8)$$

Similarly, considering node y the relation that verifies Kirchhoff's first theorem for a circuit with capacitors, expressed in the y node is obtained from (7):

$$\sum_{l_k \in y} \frac{\partial F_{E_k}}{\partial V_{y_k}} = -\sum_{l_k \in y} C_k(V_{x_k} - V_{y_k} + E_k) = -\sum_{l_k \in y} q_k = 0 \quad (9)$$

We can now state the following principle:

"In the circuits in electrostatic regime the electric charges and the potentials of the capacitors verify the Kirchhoff's theorems, and correspond to the minimum electrostatic energy of the circuit".

2.3. Magnetic circuits

The analogies between the electric and magnetic circuits as well as the similar definition of the electrostatic and magnetic energy have led to the statement of minimum energy principle of the linear magnetic circuits in quasi-stationary regime [17].

If we consider a coil having N turns, crossed by the conduction current i, situated on a linear, homogenous and isotropic magnetic material, of reluctance R_m, which has the equivalent magnetic circuit shown in Fig. 2, then Kirchhoff's second theorem for magnetic circuits in quasi-stationary regime is:
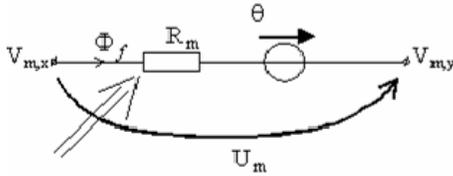
$$U_m + \theta = R_m \Phi_f \qquad (10)$$

Fig. 2. Equivalent magnetic circuit of a coil.

The magnetic energy of the circuit can be expressed in function of the magnetic potentials

$$W_m = \frac{1}{2} R_m \Phi_f^2 = \frac{(V_{m,x} - V_{m,y} + \theta)^2}{2 R_m} \qquad (11)$$

where $\theta = Ni$ is the current linkage. Similarly to the case of the electrostatic energy, the magnetic energy, having values received from the computation of the integral of a vector function, is an energy functional F_M defined into the vector space of the magnetic field S_M, with values in the real positive set R_+ ($F_M : S_M \rightarrow R_+$)

$$F_M \equiv W_M = \frac{(V_{m,x} - V_{m,y} + \theta)^2}{2 R_m} \qquad (12)$$

The energy functional F_M is positive defined, that is, $F_M(V_{m,x}, V_{m,y}) \geq 0$, whatever the values of the magnetic potentials $V_{m,x}$ and $V_{m,y}$ might be. Consequently, its extreme points are minimum points and therefore the solutions of the systems with partial derivatives

$$\frac{\partial F_M}{\partial V_{m,x}} = 0 \qquad (13)$$

and

$$\frac{\partial F_M}{\partial V_{m,y}} = 0 \qquad (14)$$

correspond to the minimum magnetic energy of the circuit.

For instance, by calculating system (13) for all the branches l_k with coils, connected to node x, we obtain a relation that verifies the Kirchhoff's first theorem for magnetic circuits expressed in node x:

$$\sum_{l_k \in x} \frac{\partial F_{M_k}}{\partial V_{m,x_k}} = \sum_{l_k \in x} \frac{V_{m,x_k} - V_{m,y_k} + \theta_k}{R_{m_k}} = \sum_{l_k \in x} \Phi_{f_k} = 0 \quad (15)$$

and a similar expression, representing a relation that verifies the Kirchhoff's first theorem for magnetic circuits expressed in node y, is found by solving the system (14):

$$\sum_{l_k \in y} \frac{\partial F_{M_k}}{\partial V_{m,x_k}} = -\sum_{l_k \in y} \frac{V_{m,x_k} - V_{m,y_k} + \theta_k}{R_{m_k}} = -\sum_{l_k \in y} \Phi_{f_k} = 0 \quad (16)$$

Hence, we can state the following principle:

"In the linear magnetic circuits in stationary regime, the linking fluxes and the magnetic potentials verify the Kirchhoff's first, respectively second theorem for magnetic circuits, and correspond to the minimum magnetic energy of the circuit".

3. Application examples

3.1. Determination of the energy functional for circuits with capacitors

Let us consider a circuit with six capacitors, respectively of capacitance C_i, for $i = 1,...6$, not charged initially, connected like in Fig. 3. By applying the Kirchhoff's second theorem we can express the electrostatic energy W_E of the system in terms of potentials V_x and V_y as:

$$W_E = \frac{1}{2}\sum_{i=1}^{5} q_i U_i = \frac{1}{2}\left[C_1(E_1 - V_x + V_y)^2 + C_2(V_x - V_z)^2 + C_3(V_x - V_t)^2 + C_4(V_y - V_z)^2 + C_5(-E_5 + V_y - V_t)^2 + C_6(V_z - V_t)^2 \right]$$

$$(17)$$

The optimisation of the energy functional F_E (5), in function of the potential V_x:

$$\frac{\partial F_E}{\partial V_x} \equiv \frac{\partial W_E}{\partial V_x} = 0 \qquad (18)$$

leads to verify the Kirchhoff's first theorem at node x

$$\frac{\partial W_E}{\partial V_x} = -C_1(E - V_x + V_y) + C_2(V_x - V_z) + C_3(V_x - V_t) = -q_1 + q_2 + q_3 = 0$$

$$(19)$$

In a similar way, if the optimisation of the energy functional of the circuit is performed relatively to the potentials V_y, V_z, or V_t, the Kirchhoff's first theorem is verified at node y, z and respectively t:

$$\frac{\partial W_E}{\partial V_y} = C_1(E - V_x + V_y) + C_4(V_y - V_z) + C_5(-E_5 + V_y - V_t) = q_1 + q_4 + q_5 = 0$$

$$(20)$$

$$\frac{\partial W_E}{\partial V_z} = -C_2(V_x - V_z) - C_4(V_y - V_z) + C_6(V_z - V_t) = -q_2 - q_4 + q_6 = 0$$

$$(21)$$

$$\frac{\partial W_E}{\partial V_t} = -C_3(V_x - V_t) - C_5(-E_5 + V_y - V_t) - C_6(V_z - V_t) = -q_3 - q_5 - q_6 = 0$$

$$(22)$$

As a conclusion, the potentials and the electric charges of the circuit are distributed in such a way to get the minimum electrostatic energy of the circuit.
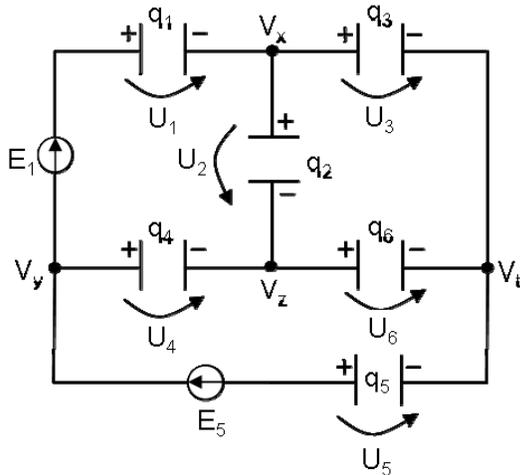
Fig. 3. Circuit with six capacitors.

3.2. Determination of the energy functional for magnetic circuits with coils

Let us consider a magnetic circuit containing four coils, connected to the nodes x and y, like in Fig. 4. By applying the Kirchhoff's second theorem for magnetic circuits, we can define the magnetic energy of the circuit W_m in terms of the magnetic potentials $V_{m,x}$ and $V_{m,y}$

$$W_M = \frac{1}{2}\sum_{i=1}^{6} R_{m_i}\Phi_{f_i}^2 = \frac{1}{2}\left[\begin{array}{l}\frac{(V_{m,z}-V_{m,x}+\theta_1)^2}{R_{m1}}+\frac{(V_{m,x}-V_{m,y}+\theta_2)^2}{R_{m2}}+\frac{(V_{m,z}-V_{m,y})^2}{R_{m3}}+\\ +\frac{(V_{m,x}-V_{m,t}+\theta_4)^2}{R_{m4}}+\frac{(V_{m,y}-V_{m,t})^2}{R_{m5}}+\frac{(V_{m,t}-V_{m,z}+\theta_6)^2}{R_{m6}}\end{array}\right] \quad (23)$$

where $R_{m,1},\dots,R_{m,6}$ are the magnetic reluctances of the branches, while $\theta_1, \theta_2, \theta_4$, and θ_6 are the current linkages of the coils.

The optimisation of the energy functional of the circuit in relation to the magnetic potential $V_{m,x}$

$$\frac{\partial F_M}{\partial V_{m,x}} \equiv \frac{\partial W_M}{\partial V_{m,x}} = 0 \quad (24)$$

leads to verify the Kirchhoff's first theorem for magnetic circuits at node x

$$\frac{\partial W_M}{\partial V_{m,x}} = -\frac{V_{mz}-V_{m,x}+\theta_1}{R_{m1}}+\frac{V_{mx}-V_{m,y}+\theta_2}{R_{m2}}+\frac{V_{mx}-V_{m,t}+\theta_4}{R_{m4}}=-\Phi_{f1}+\Phi_{f2}+\Phi_{f4}=0 \quad (25)$$

In a similar way, if the optimisation of the energy functional of the circuit is performed in relation to the potential V_y, V_z, V_t, the Kirchhoff's first theorem for magnetic circuit is verified at node y, z and respectively t:

$$\frac{\partial W_M}{\partial V_{m,y}} = -\frac{V_{mx}-V_{m,y}+\theta_2}{R_{m2}}-\frac{V_{mz}-V_{m,y}}{R_{m3}}+\frac{V_{my}-V_{m,t}}{R_{m5}}=-\Phi_{f,2}-\Phi_{f,3}+\Phi_{f5}=0 \quad (26)$$

$$\frac{\partial W_M}{\partial V_{m,z}} = \frac{V_{mz}-V_{m,x}+\theta_1}{R_{m1}}+\frac{V_{mz}-V_{m,y}}{R_{m3}}-\frac{V_{mt}-V_{m,z}+\theta_6}{R_{m6}}=\Phi_{f,1}+\Phi_{f,3}-\Phi_{f,6}=0 \quad (27)$$

$$\frac{\partial W_M}{\partial V_{m,t}} = -\frac{V_{mx}-V_{m,t}+\theta_4}{R_{m4}}-\frac{V_{my}-V_{m,t}}{R_{m5}}+\frac{V_{mt}-V_{m,z}+\theta_4}{R_{m6}}=-\Phi_{f,4}-\Phi_{f,5}+\Phi_{f,6}=0 \quad (28)$$

In conclusion, the magnetic potentials and the fluxes linking the coils in the circuit are distributed in such a way to allow the magnetic energy in the circuit to be minimal.
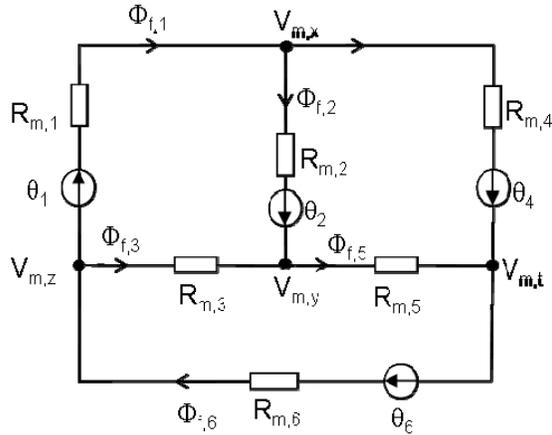

Fig. 4. Magnetic circuit with four coils.

4. Conclusions

The application of the energy functional to the analysis of electrical and magnetic circuits makes it possible to evaluate the equilibrium state, from an energy point of view, existing in the system at different time instants. Practically, to reach a balanced and stable state in the circuits, the energy in the circuit has to be minimal, providing that the state values of the circuit verify the Kirchhoff's theorems.

The formulation of the energy functionals in the Hilbert space and the determination of their extremes leads to the conclusion that the equilibrium states in the circuits represent a minimum from the energy point of view. The demonstrations presented are basically consequences of the second Kirchhoff's theorem, whereas the electric or magnetic potential used in the minimization of the energy functional verify the first Kirchhoff's theorem.

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