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Mixed mode crack propagation in advanced materials

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We consider a pre-stressed material containing a crack of a length *2a* situated in x_1x_2 – plane in mixed mode of classical fracture. We suppose that the material is unbounded and the crack faces are acted by constant normal and tangential incremental stresses. The initial applied pre-stress is in direction of the crack. Critical values of the incremental stresses and the direction of crack propagation are determined. A numerical application for a particular case of boron-epoxy composite is considered.

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1. Introduction

We consider a homogeneous, advanced material, pre-stressed and corresponding to a plane state. The advanced material represents an unbounded elastic composite containing a crack of a length *2a* situated in x_1x_3 – plane and its faces are acted by constant incremental stresses p making an angle β with Ox_2 – axis. The initial applied stress $\boldsymbol{\sigma_0}$ is in direction of the crack, as is shown in Fig. 1.

Our first aim is to determine the elastic state produced in the body using Guz's representation of incremental fields.

Our second aim is to determine the critical values of the incremental stresses and the direction of crack propagation. To do this, we use Sih's generalized fracture criterion for orthotropic elastic composite.

In the last part, using numerical analysis, we obtain for a crack in mixed mode in a pre-stressed boron - epoxy composite the critical values of the stresses which produce crack propagation and the direction of propagation.

2. Guz's representation of the incremental fields

The representation of elastic fields by complex potential in the classical case of anisotropic elastic bodies was given by Leknitskii [1]. This representation was used, for instance, by Sih and Leibowitz [2] to analyze problems concerning the existence of a crack in an anisotropic elastic solid. The results obtained by Leknitskii were generalized for the case of a pre-stressed material by Guz [3-4] who also has analysed the influence of the initial stresses on the behavior of a solid body containing cracks.

We assume that the orthotropic, initial deformed composite material is in *plane state* relative to the x_1x_2 plane. As we already know in this case, the incremental displacement field can be expressed by two real potential $\varphi^{(1)}$, $\varphi^{(2)}$, which satisfy the incremental equilibrium equations (see [3] – [6]):

$$\left(\frac{\partial^2}{\partial x_2^2}+\eta_1^2\frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial^2}{\partial x_2^2}+\eta_2^2\frac{\partial^2}{\partial x_1^2}\right)\varphi^{(2)}=0 \quad (1)$$

$$\eta^4-2A\eta^2+B=0 \quad (2)$$

$$A=\frac{\omega_{1111}\omega_{2222}+\omega_{1221}\omega_{2112}-\left(\omega_{1122}\omega_{1212}\right)^2}{2\omega_{2222}\omega_{2112}},$$
$$B=\frac{\omega_{1111}\omega_{1221}}{\omega_{2222}\omega_{2112}}. \quad (3)$$

$$\omega_{1111}=C_{11}+\sigma_0;\ \omega_{1212}=C_{66};\ \omega_{2222}=C_{22},$$
$$\omega_{1221}=C_{66}+\sigma_0;\ \omega_{1122}=C_{12};\ \omega_{2112}=C_{66} \quad (4)$$

In their turn the elastic coefficients can be expressed using the engineering constants of the material and we have the following:

$$C_{11}=\frac{1-\nu_{23}\nu_{32}}{E_1E_2D};\ C_{22}=\frac{1-\nu_{13}\nu_{31}}{E_1E_2D};$$
$$C_{12}=\frac{\nu_{12}-\nu_{32}\nu_{13}}{E_1E_2D};\ C_{66}=G_{12} \quad (5)$$

with

$$D=\frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}}{E_1E_2E_3}+$$
$$+\frac{-\nu_{21}\nu_{32}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}}{E_1E_2E_3}. \quad (6)$$

In above relations E_1, E_2, E_3 are Young's moduli in corresponding directions of material, ν_{12}, …, ν_{32} are Poisson's ratios and G_{12}, G_{23}, G_{31} are the shear moduli in the corresponding symmetry planes.

In what follows, we assume $\varphi^{(2)}(x_1, x_2) \equiv 0$ and, in order to simplify the writing, we use the notation $\varphi^{(1)}(x_1, x_2) = \varphi(x_1, x_2)$.

According to (1), $\varphi = \varphi(x_1, x_2)$ must satisfy equation

$$\left(\frac{\partial^2}{\partial x_2^2} + \eta_1^2 \frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial^2}{\partial x_2^2} + \eta_2^2 \frac{\partial^2}{\partial x_1^2}\right)\varphi = 0, \; \eta_1^2 \neq \eta_2^2. \quad (7)$$

According to Guz's representation (see [3] – [5]), the incremental displacement fields u_1 and u_2 are expressed in terms of $\varphi = \varphi^{(1)}$ by the relations

$$u_1 = -(\omega_{1122} + \omega_{1212})\varphi_{,12},$$
$$u_2 = \omega_{1111}\varphi_{,11} + \omega_{2112}\varphi_{,22}. \quad (8)$$

Let us introduce now the quantities ν_1 and ν_2 defined by

$$\nu_1 = -\eta_1^2, \nu_2 = -\eta_2^2. \quad (9)$$

Now, the differential equation (7) becomes

$$\left(\frac{\partial^2}{\partial x_2^2} - \nu_1 \frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial^2}{\partial x_2^2} - \nu_2 \frac{\partial^2}{\partial x_1^2}\right)\varphi = 0. \quad (10)$$

Also, from (2) and (9), it follows that ν_1 and ν_2 satisfy the following algebraic equation

$$\nu^2 + 2A\nu + B = 0. \quad (11)$$

Hence,

$$\nu_1 = -A - \sqrt{A^2 - B},$$
$$\nu_2 = -A + \sqrt{A^2 - B}. \quad (12)$$

Also, it can be seen that the differential equation (10) can be written in the following equivalent form:

$$\left(\frac{\partial}{\partial x_2} - \sqrt{\nu_1}\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} + \sqrt{\nu_1}\frac{\partial}{\partial x_1}\right)\cdot$$
$$\cdot\left(\frac{\partial}{\partial x_2} - \sqrt{\nu_2}\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} + \sqrt{\nu_2}\frac{\partial}{\partial x_1}\right)\varphi = 0 \quad (13)$$

Let us introduce now the parameters μ_1^2 and μ_2^2 defined by equations

$$\mu_1^2 = \nu_1, \mu_2^2 = \nu_2. \quad (14)$$

From (11), we can conclude that μ_1 and μ_2 satisfy the algebraic equation

$$\mu^4 + 2A\mu^2 + B = 0. \quad (15)$$

We assume that the initial deformed equilibrium configuration of the body is locally stable. We can conclude that the equation (15) cannot have real roots (see [5]). Consequently, from (14) we can conclude that the roots ν_1 and ν_2 must satisfy one of the following two conditions:

(1) $\operatorname{Im}\nu_j \neq 0$ or

(2) $\operatorname{Im}\nu_j = 0$ and $\operatorname{Re}\nu_j < 0, \; j = 1,2.$ (16)

We denote by μ_1, μ_2, μ_3, μ_4 the *complex roots* of the equation (15). These roots are determined by

If $\operatorname{Im}\nu_j \neq 0$, we have $\nu_1 = \overline{\nu_2}$ and we take

$$\mu_1 = \sqrt{\nu_1}, \mu_2 = -\sqrt{\nu_2},$$
$$\mu_3 = \sqrt{\nu_2} = \overline{\mu_1}, \mu_4 = -\sqrt{\nu_1} = \overline{\mu_2}. \quad (17)$$

If $\operatorname{Im}\nu_j = 0$ and $\operatorname{Re}\nu_j < 0$, we take

$$\mu_1 = \sqrt{\nu_1}, \mu_2 = \sqrt{\nu_2},$$
$$\mu_3 = -\sqrt{\nu_1} = \overline{\mu_1}, \mu_4 = -\sqrt{\nu_2} = \overline{\mu_2}. \quad (18)$$

Now, we can see that equation (13) can be expressed in the following equivalent form:

$$\left(\frac{\partial}{\partial x_2} - \mu_1\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - \overline{\mu_1}\frac{\partial}{\partial x_1}\right)\cdot$$
$$\cdot\left(\frac{\partial}{\partial x_2} - \mu_2\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - \overline{\mu_2}\frac{\partial}{\partial x_1}\right)\varphi = 0,$$
$$\mu_1 \neq \mu_2. \quad (19)$$

We introduce now the independent complex variables

$$\overline{z_1} = x_1 + \mu_1 x_2, \overline{z_2} = x_1 + \mu_2 x_2. \quad (20)$$

From these relations, we get

$$\overline{z_1} = x_1 + \overline{\mu_1} x_2, \overline{z_2} = x_1 + \overline{\mu_2} x_2. \quad (21)$$

Since $\mu_1 \neq \mu_2$, we can see now that the differential equation (19) can be expressed in the following equivalent form:

$$\frac{\partial^4 \varphi}{\partial z_1 \partial \overline{z_1} \partial z_2 \partial \overline{z_2}} = 0 . \tag{22}$$

The general solution of this equation is

$$\begin{aligned} \varphi = \varphi(x_1, x_2) = f_1(z_1) + g_1(\overline{z_1}) + \\ + f_2(z_2) + g_2(\overline{z_2}). \end{aligned} \tag{23}$$

where $f_j = f_j(z_j)$ and $g_j = g_j(\overline{z_j})$, $j = 1,2$, are arbitrary analytic functions of the complex variables z_j and $\overline{z_j}$, respectively.

We recall now that $\varphi = \varphi(x_1, x_2)$ is a *real valued* function. Hence, we must have $g_j(\overline{z_j}) = \overline{f_j(z_j)}$, $j = 1,2$.

Thus, we can conclude that the real displacement potential $\varphi = \varphi(x_1, x_2)$, satisfying the differential equation (7) can be expressed in terms of two arbitrary analytic functions $f_1 = f_1(z_1)$ and $f_2 = f_2(z_2)$, by the following relation due to Guz [3]:

$$\begin{aligned} \varphi = \varphi(x_1, x_2) = f_1(z_1) + \overline{f_1(z_1)} + f_2(z_2) + \overline{f_2(z_2)} \\ = 2\,\mathrm{Re}\{f_1(z_1) + f_2(z_2)\} \end{aligned} \tag{24}$$

Let us introduce now the analytical functions

$$F_j(z_j) = (\omega_{1122} + \omega_{1212}) f_j(z_j),\ j = 1,2 . \tag{25}$$

Thus, from (18), we get

$$\begin{aligned} \varphi = \varphi(x_1, x_2) = \\ = 2(\omega_{1122} + \omega_{1212})^{-1}\,\mathrm{Re}\{F_1(z_1) + F_2(z_2)\}. \end{aligned} \tag{26}$$

We introduce the functions $\Phi_j = \Phi_j(z_j)$, $j = 1,2$, by the following rule:

$$\Phi_j(z_j) = u_j B_j (\omega_{1122} + \omega_{1212})^{-1} F_j'''(z_j), \tag{27}$$

where

$$\begin{aligned} B_j &= \omega_{2222}\omega_{2112}\mu_j^2 + \omega_{1111}\omega_{2222} - \\ &- \omega_{1122}(\omega_{1122} + \omega_{1212}) \\ &= -\omega_{1111}\omega_{1221}\mu_j^{-2} - \omega_{2112}\omega_{1221} + \\ &+ \omega_{1212}(\omega_{1122} + \omega_{1212}). \end{aligned} \tag{28}$$

In order to obtain the second expression of B_j, we have used the fact that μ_j satisfy the algebraic equation (15), A and B being given by the relation (3).

After long but elementary computations, we get the representation of the incremental fields by two arbitrary analytic complex potentials $\Phi_j = \Phi_j(z_j)$ and their derivatives $\Psi_j = \Psi_j(z_j)$, $j = 1,2$:

$$\theta_{22} = 2\,\mathrm{Re}\{\Psi_1(z_1) + \Psi_2(z_2)\}, \tag{29}$$

$$\theta_{21} = -2\,\mathrm{Re}\{a_1\mu_1\Psi_1(z_1) + a_2\mu_2\Psi_2(z_2)\}, \tag{30}$$

$$a_j = \frac{\omega_{2112}\omega_{1122}\mu_j^2 - \omega_{1111}\omega_{1212}}{B_j\mu_j^2}, \tag{31}$$

$$\theta_{12} = -2\,\mathrm{Re}\{\mu_1\Psi_1(z_1) + \mu_2\Psi_2(z_2)\}, \tag{32}$$

$$\theta_{11} = 2\,\mathrm{Re}\{a_1\mu_1^2\Psi_1(z_1) + a_2\mu_2^2\Psi_2(z_2)\}, \tag{33}$$

$$u_1 = 2\,\mathrm{Re}\{b_1\Phi_1(z_1) + b_2\Phi_2(z_2)\}, \tag{34}$$

$$b_j = -\frac{\omega_{1122} + \omega_{1212}}{B_j}, \tag{35}$$

$$u_2 = 2\,\mathrm{Re}\{c_1\Phi_1(z_1) + c_2\Phi_2(z_2)\}, \tag{36}$$

$$c_j = \frac{\omega_{2112}\mu_j^2 + \omega_{1111}}{B_j\mu_j}. \tag{37}$$

We denoted by $\Psi_j(z_j)$ the derivates of $\Phi_j(z_j)$, $j = 1,2$, *i.e.*

$$\Psi_j(z_j) = \Phi_j(z_j) = \frac{d\Phi_j(z_j)}{dz_j},\ j = 1,2 . \tag{38}$$

It can be shown that the parameters μ_j, $j = 1,2$ satisfy the relations

$$\mathrm{Im}(\mu_1\mu_2) = 0 \text{ and } \mathrm{Re}(\mu_1 + \mu_2) = 0 . \tag{39}$$

We assume that the parameters μ_j, $j = 1,2$ are different, *i.e.*

$$\mu_1 \neq \mu_2 . \qquad (40)$$

The expressions of the complex potentials $\Psi_j(z_j)$, $j = 1,2$ corresponding to our mixed mode are

$$\Psi_1(z_1) = \frac{a_2\mu_2 K_I + K_{II}}{2\sqrt{2\pi r}\Delta} \cdot \frac{1}{\chi_1(\varphi)},$$
$$\Psi_2(z_2) = -\frac{a_1\mu_1 K_I + K_{II}}{2\sqrt{2\pi r}\Delta} \cdot \frac{1}{\chi_2(\varphi)}. \qquad (41)$$

where

$$\Delta = a_2\mu_2 - a_1\mu_1,$$

$$\chi_j(\varphi) = \sqrt{\cos\varphi + \mu_j \sin\varphi} . \qquad (42)$$

and

$$K_I = p\sin^2\beta\sqrt{\pi a},$$
$$K_{II} = p\sin\beta\cos\beta\sqrt{\pi a} . \qquad (43)$$

are the stress intensity factors corresponding to the first respectively second mode of fracture for an applied load $p > 0$.

3. Sih's generalized fracture criterion for a mixed fracture mode

Let us denote by *W the incremental strain energy density, i.e.*

$$W = \frac{1}{2}\theta_{k,l}u_{l,k},\ k,l = 1,2, \qquad (39)$$

where

$$u_{l,k} = \frac{\partial u_l}{\partial x_k} . \qquad (40)$$

Let r and φ being the radial distance from the crack tip and the angle between radial direction and the line ahead the crack, as in Fig. 1. After long manipulations we obtain that near the considered crack tip the strain energy has a singular part as well a regular part, *i.e.*

$$W(r,\mu) = \frac{S(\varphi)}{r} + a\ regular\ part . \qquad (41)$$

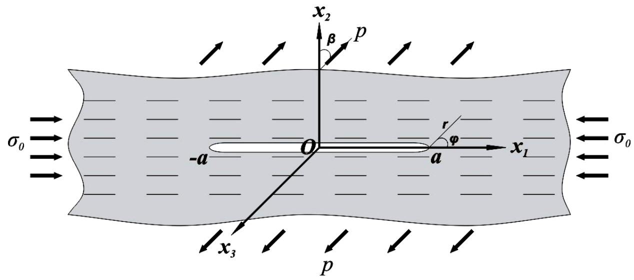

Fig. 1. Mixed mode crack propagation.

Here $S(\varphi)$ is *Sih's incremental strain energy density factor* and is given by

$$S(\varphi) = \frac{(a_1\mu_1 K_I + K_{II})(a_2\mu_2 K_I + K_{II})}{4\pi} s_m(\varphi) \qquad (42)$$

and

$$\begin{aligned} s_m(\varphi) = &\operatorname{Re}\left[\frac{a_1a_2\mu_1\mu_2}{\Delta}\left(\frac{\mu_1}{\chi_1(\varphi)} - \frac{\mu_2}{\chi_2(\varphi)}\right)\right]\cdot \\ &\cdot\operatorname{Re}\left[\frac{1}{\Delta}\left(\frac{a_2\mu_2 b_1}{\chi_1(\varphi)} - \frac{a_1\mu_1 b_2}{\chi_2(\varphi)}\right)\right] - \\ &-\operatorname{Re}\left[\frac{a_1a_2\mu_1\mu_2}{\Delta}\left(\frac{1}{\chi_1(\varphi)} - \frac{1}{\chi_2(\varphi)}\right)\right]\cdot \\ &\cdot\operatorname{Re}\left[\frac{1}{\Delta}\left(\frac{a_2 b_1}{\chi_1(\varphi)} - \frac{a_1 b_2}{\chi_2(\varphi)}\right)\right] - \\ &-\operatorname{Re}\left[\frac{\mu_1\mu_2}{\Delta}\left(\frac{a_2}{\chi_1(\varphi)} - \frac{a_1}{\chi_2(\varphi)}\right)\right]\cdot \\ &\cdot\operatorname{Re}\left[\frac{1}{\Delta}\left(\frac{a_2 c_1\mu_2}{\chi_2(\varphi)} - \frac{a_1 c_1\mu_1}{\chi_1(\varphi)}\right)\right] + \\ &+\operatorname{Re}\left[\frac{1}{\Delta}\left(\frac{a_2\mu_2}{\chi_1(\varphi)} - \frac{a_1\mu_1}{\chi_2(\varphi)}\right)\right]\cdot \\ &\cdot\operatorname{Re}\left[\frac{1}{\Delta}\left(\frac{a_2 c_1}{\chi_1(\varphi)} - \frac{a_1 c_2}{\chi_2(\varphi)}\right)\right]. \end{aligned} \qquad (43)$$

We generalize the Sih's fracture criterion (see [2]) for orthotropic or pre-stressed elastic materials, assuming that:

H1: Crack propagation will start in a radial direction φ_c along with the incremental strain energy density $S(\varphi)$ is a minimum, *i.e.*

$$\frac{dS}{d\varphi}(\varphi_c) = 0, \frac{d^2S}{d\varphi^2}(\varphi_c) > 0. \quad (44)$$

H2: The critical intensity

$$S_c = S_{\min} = S(\varphi_c). \quad (45)$$

governs the onset of the crack propagation and it represents a material constant independent the crack geometry, loading and initial stress.

Using (42) and (45) we get for the incremental stress P_c for which the crack will start to propagate at critical direction φ_c the following equation:

$$aP_c^2 = \frac{4S_c}{s_m(\varphi_c)}. \quad (46)$$

Once S_c is known, the relation (46) can be used to get P_c.

4. Numerical results and conclusions.

In this section we shall consider the case of a boron-epoxy composite material characterized by the following parameters:

$$E_1 = 190\,GPa, \quad E_2 = E_3 = 10\,GPa,$$
$$G_{12} = 7\,GPa, \quad G_{13} = G_{23} = 6\,GPa,$$
$$\nu_{12} = 0.3, \quad \nu_{13} = \nu_{23} = 0.2. \quad (47)$$

For a composite material is a critical value σ_0^c of the initial applied stress σ_0 for which such than when σ_0 tends to σ_0^c, the incremental stress P_c converges to zero, and it is given by:

$$\sigma_0^c = -G_{12}\left\{1 - \frac{G_{12}}{E_1E_2}(1 - \nu_{13}\nu_{31})\right\} < 0. \quad (48)$$

For our composite material one gets the following value:

$$\sigma_0^c = -6.839. \quad (49)$$

So, in our study we shall consider $\sigma_0 \in (-6.839, 0]$. From our numerical analysis we observe that:

– the strain energy density $s_m(\varphi)$ depends in a very small manner by σ_0 and in this case the propagation angle is in a neighborhood of 70°, as in Fig. 2.

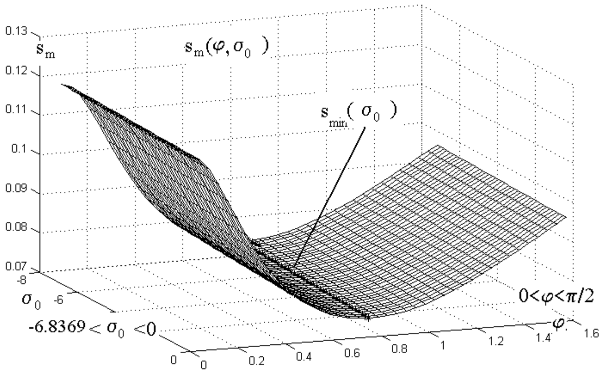

Fig. 2. Representation of the function s_m versus φ and σ_0.

– the critical intensity density factor S_c decreases when β decreases and the initial pre-stressed σ_0 doesn't play an important role in this case due to the fact that $\sigma_0 < E_2 << E_1$, as in Fig. 3.

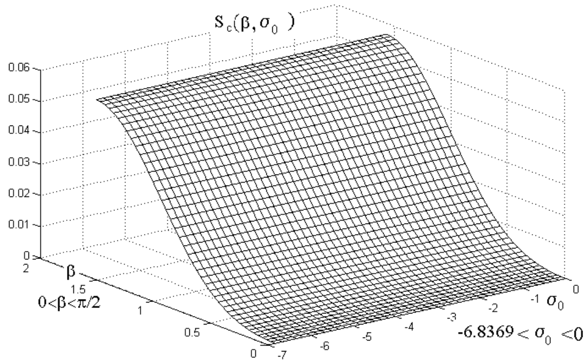

Fig. 3. Representation of the critical intensity of the strain energy density factor S versus β and σ_0.

– for different values of the angle β we obtain that the minimum of $S(\varphi, \sigma_0)$ is obtained for a critical value φ_c, as in Fig. 4.

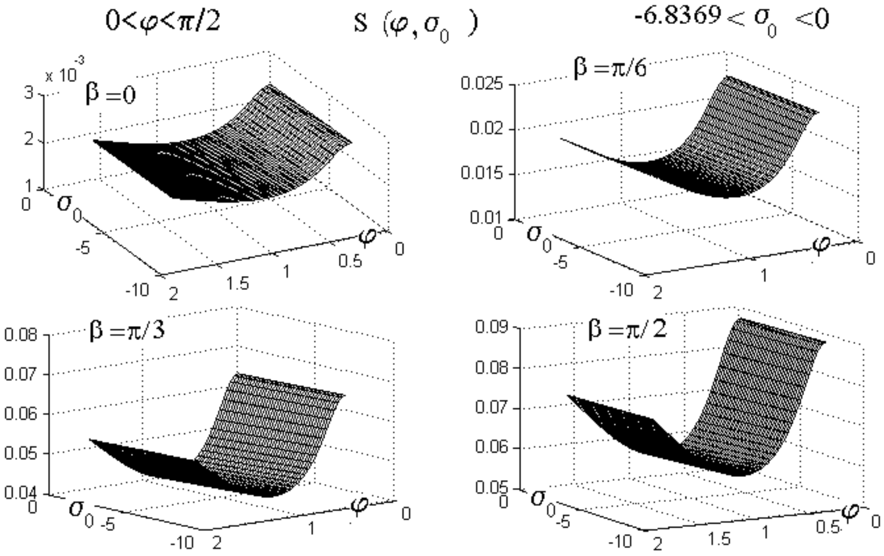

Fig. 4. Representation of the the strain energy density factor S versus φ *and* σ_0 *for different angles* β.

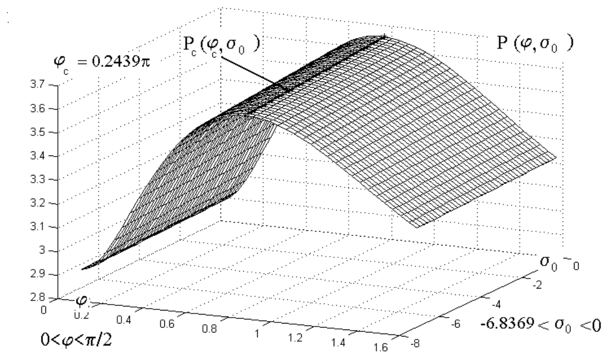

Fig. 5. Incremental stress P versus φ *and* σ_0.

When $\beta = 0$ and the material is unpre-stressed $\sigma_0 = 0$, we observe that $\varphi_c = 0$, an well-known result, *i.e.* in the Mode I of fracture, the crack will propagate along its line.

– using eqs. (46) we found the critical incremental stress P_c which produces the initialization of crack propagation, as in Fig. 5.

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