New boundedness results for solutions of second order non-autonomous delay differential equations

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We study the boundedness of solutions of some second order non-autonomous delay differential equations by the Liapunov functional approach. We establish three new results which include sufficient conditions for the solutions of the equations considered to be bounded. By this work, we improve some boundedness results in the literature, which were obtained on certain second order ordinary differential equations without delay, to the boundedness of the solutions of some second order non-autonomous delay differential equations.

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1. Introduction and main results

In applied science, second order nonlinear differential equations with and without delay are used to model some practical problems in biology, chemistry, physics, mechanics, electronics, engineering, economy, control theory, medicine, atomic energy, information theory, etc. (see, for example, the book of Ahmad and Rama Mohana Rao [1] and the reference thereof).

In 1979, Graef [6] considered the second order nonlinear differential equation without delay,

$$(a(t)x')' + h(t,x,x') + q(t)f(x)g(x') = e(t,x,x'). \quad (1)$$

The author established three theorems which include some sufficient conditions and guarantee that all solutions of Eq. (1) are bounded.

In this paper, instead of Eq. (1), we consider the second order non-autonomous delay differential equations of the form

$$\begin{aligned}(a(t)x')' + h(t,x(t),x(t-r),x'(t),x'(t-r))x'(t) + q(t)g(x'(t))f(x(t-r))\\ = e(t,x(t),x(t-r),x'(t),x'(t-r)). \quad (2)\end{aligned}$$

We write Eq. (2) in system form as $x' = y,$

$$\begin{aligned}y' = -\frac{1}{a(t)}[a'(t)y + h(t,x,x(t-r),y,y(t-r))y + q(t)f(x)g(y)]\\ + \frac{q(t)}{a(t)}g(y)\int_{t-r}^{t} f'(x(s))y(s)ds\\ + \frac{1}{a(t)}e(t,x,x(t-r),y,y(t-r)), \quad (3)\end{aligned}$$

where $x(t)$ and $y(t)$ are abbreviated as x and y, respectively, and we assume that $a,\ q:[t_0,\infty) \to \Re,$ $t_0 \ge 0,$ $f,\ g:\Re \to \Re$ and $h, e:[t_0,\infty)\times\Re^4 \to \Re$ are continuous, and $a,\ q$ differentiable, $a(t) > 0,$ $q(t) > 0$ and $g(x') > 0.$

Our motivation comes especially from the paper of Graef [6]. The principal aim of this paper is to improve the boundedness results established in Graef [6] for Eq. (1) to the boundedness of solutions of nonlinear delay differential Eq. (2). By defining three new Liapunov functionals, we prove the results established here and also follow a similar way as indicated in [6] for verifying our main results.

It should be noted that prototypes for studying Eq. (1) and Eq. (2) are the well known autonomous equations of van der Pol and Liénard (see Reissig et al. [13], Graef [5]) and the non-autonomous Emden-Fowler equation (see Coffman and Wong [4], Mustafa and Tunç [12] and the references thereof). For a survey of some results of this type and the others, in particular, we refer the reader to the papers of Baker [2], Burton and Grimmer [3], Graef and Spikes [7, 8, 9], Jin [10], Kroopnick [11], Saker [14], Sun [15], Tunç [16-24], Tunç and Sevli [25], C. Tunç and E. Tunç [26] and the references contained therein.

The results presented here differ in some respects from those usually found in the literature. Namely, to the best of our knowledge, there is no published paper in recent literature on the boundedness of solutions of the second order non-autonomous delay differential equations of the form (2), when $a(t) \ne 1,$ $q(t) \ne 1$ and $g(x') \ne 1.$ This is to say that we did not find any work on the boundedness of solutions in the literature, which based on the results of Graef [6]. In addition, we allow for large negative damping and do not require that forcing term $e(t,x,x(t-r),y,y(t-r))$ be small.

Let $q'(t)_+ = \max\{q'(t),0\}$ and $q'(t)_- = \max\{-q'(t),0\}$ so that $q'(t) = q'(t)_+ - q'(t)_-$.

Assume that there are nonnegative continuous functions $r_1, r_2, w:[t_0,\infty)\to\Re$ such that

$$|e(t,x,x(t-r),y,y(t-r))| \le r_1(t) + r_2(t)|y|, \quad (4)$$

and there are positive constants β, M, k, a_2, q_2 and α such that

$$\beta - w(t) \le h(t,x,x(t-r),y,y(t-r)), \quad (5)$$

$$\frac{y^2}{g(y)} \le MG(y) \text{ for } |y| \ge k, \quad (6)$$

$$0 < f'(x) \le \alpha, \quad (7)$$

$$\int_{t_0}^{\infty} \frac{a'(s)_-}{a(s)} ds < \infty, \quad a(t) \le a_2, \quad (8)$$

$$\int_{t_0}^{\infty} \frac{q'(s)_-}{q(s)} ds < \infty, \quad q(t) \le q_2, \quad (9)$$

and

$$\int_{t_0}^{\infty} w(s)ds < \infty. \quad (10)$$

Our first result is the following theorem.

Theorem 1. If in addition to conditions (4)-(10), we have

$$\int_0^x f(s)ds \to \infty \text{ as } |x|\to\infty, \; c_1 \ge g(x') \ge c > 0,$$

(where c and c_1 are some constants),

$$\int_{t_0}^{\infty} r_2(s)ds < \infty \text{ and } \int_{t_0}^{\infty} \frac{r_1(s)}{\sqrt{q(s)}} ds < \infty,$$

then all solutions of Eq. (2) defined by the initial function

$$x(t) = \phi(t), \; x'(t) = \phi'(t)$$

are bounded for all $t \ge t_0$, where $\phi \in C^1([t_0 - r, t_0], \Re)$, provided

$$r < \frac{a_1\beta}{a_2 q_2 c_1 \alpha}.$$

Proof. Since $\int_0^x f(s)ds \to \infty$ as $|x|\to\infty$, $\int_0^x f(s)ds$ is bounded from below, say $\int_0^x f(s)ds \ge -K$ for some $K>0$. Note also that conditions (8) and (9) imply that $a(t) \ge a_1 > 0$ and $q(t) \ge q_1 > 0$, where a_1 and q_1 are some constants.

Define the Liapunov functional $V = V(t, x_t, y_t)$,

$$V(t,x_t,y_t) = \frac{1}{a(t)}\int_0^x f(s)ds + \frac{K}{a(t)} + \frac{1}{q(t)}\int_0^y \frac{s}{g(s)}ds + \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds,$$

where λ_1 is a positive constant to be determined later.

Let $(x,y) = (x(t),y(t))$ be a solution of (3). Differentiating the Liapunov functional $V(t,x_t,y_t)$ along this solution, we get

$$\frac{d}{dt}V(t,x_t,y_t) = -\frac{a'(t)}{a^2(t)}\left(\int_0^x f(s)ds + K\right) - \frac{q'(t)}{q^2(t)}\int_0^y \frac{s}{g(s)}ds - \frac{a'(t)}{a(t)q(t)g(y)}y^2 - \frac{h(t,x,x(t-r),y,y(t-r))}{a(t)q(t)g(y)}y^2 + \frac{e(t,x,x(t-r),y,y(t-r))y}{a(t)q(t)g(y)} + \frac{y}{a(t)}\int_{t-r}^{t} f'(x(s))y(s)ds + \lambda_1 r y^2 - \lambda_1 \int_{t-r}^{t} y^2(s)ds.$$

In the light of the assumptions of $0 < f'(x) \le \alpha$, $a_1 \le a(t)$ and the inequality $2|uv| \le u^2 + v^2$, it follows

$$\frac{y}{a(t)}\int_{t-r}^{t} f'(x(s))y(s)ds \le \frac{|y|}{a(t)}\int_{t-r}^{t} f'(x(s))|y(s)|ds \le \frac{\alpha r}{2a_1}y^2 + \frac{\alpha}{2a_1}\int_{t-r}^{t} y^2(s)ds.$$

By using the assumptions of the theorem and the foregoing inequality, we obtain

$$\frac{d}{dt}V(t,x_t,y_t) \le$$
$$\frac{a'(t)_-}{a^2(t)}(\int_0^x f(s)ds + K) + \frac{q'(t)_-}{q^2(t)}\int_0^y \frac{s}{g(s)}ds$$
$$-\frac{\beta}{a_2 q_2 c_1}y^2$$
$$+\frac{a'(t)_-}{a(t)q(t)g(y)}y^2 + \frac{w(t)y^2}{a(t)q(t)g(y)}$$
$$+\frac{r_1(t)|y|}{a(t)q(t)g(y)}$$
$$+\frac{r_2(t)y^2}{a(t)q(t)g(y)}$$
$$+\left(\frac{\alpha r}{2a_1}+\lambda_1 r\right)y^2 + \left(\frac{\alpha}{2a_1}-\lambda_1\right)\int_{t-r}^t y^2(s)ds.$$

Let $\lambda_1 = \frac{\alpha}{2a_1}$. Hence, we have

$$\frac{d}{dt}V(t,x_t,y_t) \le$$
$$\frac{a'(t)_-}{a^2(t)}(\int_0^x f(s)ds + K) + \frac{q'(t)_-}{q^2(t)}\int_0^y \frac{s}{g(s)}ds$$
$$+\frac{a'(t)_-}{a(t)q(t)g(y)}y^2 + \frac{w(t)y^2}{a(t)q(t)g(y)}$$
$$+\frac{r_1(t)|y|}{a(t)q(t)g(y)}$$
$$+\frac{r_2(t)y^2}{a(t)q(t)g(y)} - \left(\frac{\beta}{a_2 q_2 c_1} - \frac{\alpha r}{a_1}\right)y^2.$$

Using the estimate $r < \frac{a_1\beta}{a_2 q_2 c_1 \alpha}$, it follows

$$\frac{d}{dt}V(t,x_t,y_t) \le$$
$$\frac{a'(t)_-}{a^2(t)}(\int_0^x f(s)ds + K) + \frac{q'(t)_-}{q^2(t)}\int_0^y \frac{s}{g(s)}ds$$
$$+\frac{a'(t)_-}{a(t)q(t)g(y)}y^2 + \frac{w(t)y^2}{a(t)q(t)g(y)}$$
$$+\frac{r_1(t)|y|}{a(t)q(t)g(y)} + \frac{r_2(t)y^2}{a(t)q(t)g(y)}.$$

If $\frac{|y|}{\sqrt{q(t)}} \ge 1,$ then $\frac{|y|}{\sqrt{q(t)}} \le \frac{y^2}{q(t)}+1,$ and if $\frac{|y|}{\sqrt{q(t)}} \le 1,$ then $\frac{|y|}{\sqrt{q(t)}} \le \frac{y^2}{q(t)}+1.$ Also, for $|y| \le k,$ $\frac{y^2}{g(y)} \le N$ for some $N > 0,$ so $\frac{y^2}{g(y)} \le N + MG(y)$ for all $y.$ Hence,

$$\frac{d}{dt}V(t,x_t,y_t) \le$$
$$\left[(M+1)\frac{a'(t)_-}{a(t)} + \frac{q'(t)_-}{q(t)} + \frac{M}{a_1}w(t) + \frac{M}{a_1}r_2(t)\right]V$$
$$+\frac{Na'(t)_-}{q_1 a(t)} + \frac{N}{q_1 a_1}w(t) + \frac{N}{q_1 a_1}r_2(t) + \frac{1}{ca_1}\frac{r_1(t)}{\sqrt{q(t)}}$$
$$+\frac{r_1(t)}{a_1\sqrt{q^3(t)}}\frac{y^2}{g(y)}$$
$$\le$$
$$\left[(M+1)\frac{a'(t)_-}{a(t)} + \frac{q'(t)_-}{q(t)} + \frac{M}{a_1}w(t) + \frac{M}{a_1}r_2(t) + \frac{M}{a_1}\frac{r_1(t)}{\sqrt{q(t)}}\right]V$$
$$+\frac{N}{q_1}\left[\frac{a'(t)_-}{a(t)} + \frac{1}{a_1}w(t) + \frac{1}{a_1}r_2(t)\right] + \frac{1}{ca_1}\frac{r_1(t)}{\sqrt{q(t)}}$$
$$+\frac{N}{a_1 q_1}\frac{r_1(t)}{\sqrt{q(t)}}$$
$$= k_1(t)V + k_2(t),$$

where

$$k_1(t) = (M+1)\frac{a'(t)_-}{a(t)} + \frac{q'(t)_-}{q(t)} + \frac{M}{a_1}w(t) + \frac{M}{a_1}r_2(t) + \frac{M}{a_1}\frac{r_1(t)}{\sqrt{q(t)}},$$

$$k_2(t) = \frac{N}{q_1}\left[\frac{a'(t)_-}{a(t)} + \frac{1}{a_1}w(t) + \frac{1}{a_1}r_2(t)\right] + \frac{1}{ca_1}\frac{r_1(t)}{\sqrt{q(t)}}$$
$$+\frac{N}{a_1 q_1}\frac{r_1(t)}{\sqrt{q(t)}}.$$

Integrating the last above inequality from 0 to $t,$ it follows

$$V(t,x_t,y_t) \le V(0,x_0,y_0)$$
$$+\int_0^t k_1(s)V(s,x_s,y_s)ds + \int_0^t k_2(s)ds.$$

Applying the Gronwall-Reid-Bellman inequality, (see Ahmad and Rama Mohana Rao [1]), and observing $\int_0^\infty k_1(s)ds < \infty$ and $\int_0^\infty k_2(s)ds < \infty$, we immediately obtain that $V(t, x_t, y_t)$ is bounded. Further, since $a(t) \le a_2$, we have that $F(x(t))$ is bounded from which we have that $x(t)$ is bounded for all $t \ge t_0 \ge 0$. This completes the proof of Theorem 1.

Our second result is the following theorem.

Theorem 2. Suppose conditions (5)-(8) and (10) hold, there is a continuous function $r : [t_0, \infty) \to \Re$ and a constant $d > 0$ such that

$$c_1 \ge g(x') > 0,$$

$$|e(t, x, x(t-r), y, y(t-r))y| \le \frac{q(t)g(y)}{r^d(t)},$$

$$\int_{t_0}^\infty \frac{r'(s)_-}{r(s)} ds < \infty, \quad \int_{t_0}^\infty \frac{1}{r^d(s)} ds < \infty, \quad H(t) = \frac{r(t)}{q(t)}$$

is bounded,

and

$$\int_{t_0}^\infty \frac{H'(s)_-}{H(s)} ds < \infty.$$

If $\int_0^x f(s)ds \to \infty$ as $|x| \to \infty$,

then all solutions of Eq. (2) defined by the initial function

$$x(t) = \phi(t), \; x'(t) = \phi'(t)$$

are bounded for all $t \ge t_0$, where $\phi \in C^1([t_0 - r, t_0], \Re)$, provided $r < \dfrac{\beta r_1 a_1}{c_1 q_2 r_2 a_2 \alpha}$.

Proof. Again we have $\int_0^x f(s)ds \ge -K$ for some $K > 0$. Define the Liapunov functional

$$V_1(t, x_t, y_t) =$$

$$\frac{1}{a(t)H(t)}\left(\int_0^x f(s)ds + K\right) + \frac{1}{r(t)}\int_0^y \frac{s}{g(s)} ds + \lambda_2 \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds,$$

where λ_2 is a positive constant to be determined later. Let $(x, y) = (x(t), y(t))$ be a solution of (3). Differentiating the Liapunov functional $V_1(t, x_t, y_t)$ along this solution, we get

$$\frac{d}{dt}V_1(t, x_t, y_t) =$$

$$-\frac{[a'(t)H(t) + a(t)H'(t)]}{a^2(t)H^2(t)}\left(\int_0^x f(s)ds + K\right) - \frac{r'(t)}{r^2(t)}\int_0^y \frac{s}{g(s)} ds$$

$$-\frac{a'(t)}{r(t)a(t)g(y)} y^2 - \frac{h(t, x, x(t-r), y, y(t-r))}{r(t)a(t)g(y)} y^2$$

$$+ \frac{e(t, x, x(t-r), y, y(t-r))y}{a(t)r(t)g(y)}$$

$$+ \frac{q(t)y}{r(t)a(t)} \int_{t-r}^t f'(x(s))y(s)ds$$

$$+ \lambda_2 r y^2 - \lambda_2 \int_{t-r}^t y^2(s)ds.$$

Making use of the assumptions of Theorem 2, it follows

$$\frac{q(t)y}{r(t)a(t)} \int_{t-r}^t f'(x(s))y(s)ds \le \frac{q(t)|y|}{r(t)a(t)} \int_{t-r}^t f'(x(s))|y(s)|ds$$

$$\le \frac{q_2 \alpha r}{2r_1 a_1} y^2 + \frac{q_2 \alpha}{2r_1 a_1} \int_{t-r}^t y^2(s)ds.$$

Hence, in the light of the assumption of Theorem 2, we obtain

$$\frac{d}{dt}V_1(t, x_t, y_t) \le$$

$$-\frac{[a'(t)H(t) + a(t)H'(t)]}{a^2(t)H^2(t)}\left(\int_0^x f(s)ds + K\right) + \frac{r'(t)_-}{r^2(t)}\int_0^y \frac{s}{g(s)} ds$$

$$+ \frac{a'(t)_-}{r(t)a(t)g(y)} y^2 + \frac{w(t)}{r(t)a(t)g(y)} y^2$$

$$+ \frac{q(t)}{r^{d+1}(t)a(t)} - \left[\frac{\beta}{r_1 a_1 c_1} - \left(\frac{q_2 \alpha}{2r_1 a_1} + \lambda_2\right) r\right] y^2$$

$$- \left(\lambda_2 - \frac{q_2 \alpha}{2r_1 a_1}\right) \int_{t-r}^t y^2(s)ds.$$

Let $\lambda_2 = \dfrac{q_2 \alpha}{2r_1 a_1}$. Hence, we get

$$\frac{d}{dt}V_1(t, x_t, y_t) \le$$

$$-\frac{[a'(t)H(t) + a(t)H'(t)]}{a^2(t)H^2(t)}\left(\int_0^x f(s)ds + K\right) + \frac{r'(t)_-}{r^2(t)}\int_0^y \frac{s}{g(s)} ds$$

$$+ \frac{a'(t)_-}{r(t)a(t)g(y)} y^2 + \frac{w(t)}{r(t)a(t)g(y)} y^2$$

$$+ \frac{q(t)}{r^{d+1}(t)a(t)} - \left(\frac{\beta}{r_2 a_2 c_1} - \frac{q_2 \alpha}{r_1 a_1} r\right) y^2.$$

Using the estimate $r < \dfrac{\beta r_1 a_1}{c_1 q_2 r_2 a_2 \alpha}$ and the assumptions of Theorem 2, we have

$$\frac{d}{dt}V_1(t,x_t,y_t) \le \left[\frac{H'(t)_-}{H(t)} + (M+1)\frac{a'(t)_-}{a(t)} + \frac{r'(t)_-}{r(t)} + \frac{M}{a_1}w(t)\right]V_1 + N\frac{a'(t)_-}{a(t)r_1} - N\frac{w(t)}{r_1 a_1} + \frac{1}{r^d(t)H_1 a_1}.$$

Finally, as in the proof of Theorem 1, it follows that $x(t)$ is bounded.

If we consider the special case of Eq. (2) with $g(x') \equiv 1,$ namely, we take into consideration the second order non-autonomous delay differential equation

$$(a(t)x')' + h(t,x(t),x(t-r),x'(t),x'(t-r))x'(t) + q(t)f(x(t-r)) = e(t,x(t),x(t-r),x'(t),x'(t-r)). \tag{11}$$

We write Eq. (11) in system form as

$$x' = y,$$
$$y' = -\frac{1}{a(t)}[a'(t)y + h(t,x,x(t-r),y,y(t-r))y + q(t)f(x)] + \frac{q(t)}{a(t)}\int_{t-r}^{t} f'(x(s))y(s)ds + \frac{1}{a(t)}e(t,x,x(t-r),y,y(t-r)). \tag{12}$$

Our last result is given by the following theorem.

Theorem 3. Suppose conditions (4) and (5) hold,

$$\int_{t_0}^{\infty}\frac{(a(s)q(s))'_-}{a(s)q(s)}ds < \infty,\ \int_{t_0}^{\infty}\frac{r_1(s)}{a(s)\sqrt{q(s)}}ds < \infty,$$
$$\int_{t_0}^{\infty}\frac{w(s)}{a(s)}ds < \infty,\ \int_{t_0}^{\infty}\frac{r_1(s)}{\sqrt{q(s)}}ds < \infty$$

and

$$\int_{t_0}^{\infty}\frac{r_2(s)}{a(s)}ds < \infty.$$

If $\int_0^x f(s)ds \to \infty$ as $|x| \to \infty,$ then all solutions of Eq. (11) defined by the initial function

$$x(t) = \phi(t),\ x'(t) = \phi'(t)$$

are bounded for all $t \ge t_0,$ where $\phi \in C^1([t_0 - r, t_0], \Re),$ provided that $r < \dfrac{\beta}{\alpha q_2}.$

Proof. Define the Liapunov functional

$$V_2(t,x_t,y_t) = \frac{a(t)}{q(t)}y^2 + 2(\int_0^x f(s)ds + K) + \lambda_3\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds,$$

where $K > 0$ is defined as before and λ_3 is a positive constant to be determined later.

Let $(x,y) = (x(t),y(t))$ be a solution of (12). Differentiating the Liapunov functional $V_2(t,x_t,y_t)$ along this solution, we get

$$\frac{d}{dt}V_2(t,x_t,y_t) = -\frac{a(t)q'(t)}{q^2(t)}y^2 - \frac{a'(t)}{q(t)}y^2 - \frac{2h(t,x,x(t-r),y,y(t-r))}{q(t)}y^2 + \frac{2e(t,x,x(t-r),y,y(t-r))y}{q(t)} + 2y\int_{t-r}^{t} f'(x(s))y(s)ds \quad + \lambda_3 r y^2 - \lambda_3\int_{t-r}^{t} y^2(s)ds.$$

In the light of the assumptions of $0 < f'(x) \le \alpha$ and the inequality $2|uv| \le u^2 + v^2,$ it follows

$$2y\int_{t-r}^{t} f'(x(s))y(s)ds \le 2|y|\int_{t-r}^{t} f'(x(s))|y(s)|ds \le \alpha r y^2 + \alpha\int_{t-r}^{t} y^2(s)ds.$$

By using the assumptions of the theorem and the foregoing inequality, we obtain

$$\frac{d}{dt}V_2(t,x_t,y_t) \le -\frac{(a(t)q(t))'}{q^2(t)} + \frac{2w(t)}{q(t)}y^2 - \frac{2\beta}{q_2}y^2 + \frac{2r_1(t)|y|}{q(t)} + \frac{2r_2(t)y^2}{q(t)} + (\alpha+\lambda_3)ry^2 + (\alpha-\lambda_3)\int_{t-r}^{t} y^2(s)ds.$$

Let $\lambda_3 = \alpha.$ Hence, we have

$$\frac{d}{dt}V_2(t,x_t,y_t) \le -\frac{(a(t)q(t))'}{q^2(t)}y^2 + \frac{2w(t)}{q(t)}y^2 + \frac{2r_1(t)|y|}{q(t)} + \frac{2r_2(t)y^2}{q(t)} - 2(\beta q_2^{-1} - \alpha r)y^2.$$

Using the estimate $r < \dfrac{\beta}{\alpha q_2}$, it follows

$$\frac{d}{dt}V_2(t,x_t,y_t) \le -\frac{(a(t)q(t))'}{q^2(t)}y^2 + \frac{2w(t)}{q(t)}y^2 + \frac{2r_1(t)|y|}{q(t)} + \frac{2r_2(t)y^2}{q(t)}.$$

Hence, we have

$$\frac{d}{dt}V_2(t,x_t,y_t) \le \left[\frac{(a(t)q(t))'_-}{a(t)q(t)} + \frac{2w(t)}{a(t)} + \frac{2r_1(t)}{a(t)\sqrt{q(t)}} + \frac{2r_2(t)}{a(t)}\right]V_2 + \frac{2r_1(t)}{\sqrt{q(t)}}.$$

The remainder of the proof follows as before.

References

[1] S. Ahmad; M. Rama Mohana Rao, Theory of ordinary differential equations. With applications in biology and engineering. Affiliated East-West Press Pvt. Ltd., New Delhi, 1999.
[2] J. W. Baker, J. Appl. Math. **27**, 159 (1974).
[3] T. A. Burton, R. C. Grimmer, Monatsh. Math. **74**, 211 (1970).
[4] C. V. Coffman, J. S.W. Wong, Trans. Amer. Math. Soc. **167**, 399 (1972).
[5] J. R. Graef, J. Differential Equations **12**, 34 (1972).
[6] J. R. Graef, Abh. Math. Sem. Univ. Hamburg **49**, 70 (1979).
[7] J. R. Graef, P. W. Spikes, J. Differential Equations **17**, 461 (1975).
[8] J. R. Graef, P. W. Spikes, J. Math. Anal. Appl. **62**(2), 295 (1978),
[9] J. R. Graef, P. W. Spikes, Publ. Math. Debrecen **24**(1-2), 39 (1977).
[10] Z. Jin, J. Math. Anal. Appl. **256**(2), 360 (2001).
[11] A. Kroopnick, J. Math. Math. Sci. **18**(4), 823 (1995),.
[12] O. G. Mustafa, C. Tunç, Appl. Math. Comput. **215**(8), 3076 (2009).
[13] R. Reissig, G. Sansone, R. Conti, Qualitative Theorie nichtlinearer Differentialgleichungen. (German) Edizioni Cremonese, Rome 1963.
[14] S. H. Saker,. Rocky Mountain J. Math. **36**(6), 2027 (2006).
[15] S. Sun, J. Ocean Univ. China Nat. Sci. **36**(3), 397 (2006).
[16] C. Tunç, Arab. J. Sci. Eng. Sect. A Sci. **33**(1), 83 (2008).
[17] C. Tunç, Iran. J. Sci. Technol. Trans. A Sci. **30**(2), 213 (2006).
[18] C. Tunç, J. Comput. Anal. Appl. **11**(4), 711 (2009).
[19] C. Tunç, Nonlinear Anal. Hybrid Syst. **4**(1), 85 (2010).
[20] C. Tunç, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **53(101)**(1), 61 (2010).
[21] C. Tunç, Appl. Anal. Discrete Math.**4(**2), 361 (2010).
[22] C. Tunç, J. Indones. Math. Soc.. **16**(2), 110 (2010).
[23] C. Tunç, J. Contemp. Math. Anal. **45**(3), 47 (2010).
[24**]** C. Tunç, Stability, J. Comput. Anal. Appl. **13**(6), 1067 (2011).
[25] C.Tunç, H. Şevli, J. Franklin Inst. **344**(5), 399 (2007).
[26] C. Tunç, E. Tunç, J. Franklin Inst. **344**(5), 391 (2007).

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