Normalized E-factor. A digital region descriptor for 2D and 3D space shape classification

R. SANTIAGO-MONTERO, D. A. GUTIÉRREZ-HERNÁNDEZ*, D. A. OLIVARES-VERA, V. ZAMUDIO
Tecnológico Nacional de México. Instituto Tecnológico de León. División de Estudios de Posgrado e Investigación, Avenida Tecnológico s/n, Industrial Julián de Obregón, C.P. 37290 León, Guanajuato, México

Nowadays, a goodness digital compactness measure is necessary in computer vision, shape analysis and computer medical diagnosis process where digital picture are used widely. We introduce a compactness measure called Normalized E-Factor which shows as a measure robust to translations, rotations and scale-changes and that it satisfies the set of criteria for a good compactness measure. Through a series of experiments, we show that the Normalized E-Factor is useful for shape description, measuring digital compactness with or without holes and that it overcomes some drawbacks that present several compactness measures over digital space.

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1. Introduction

Shape classification is an active research area in pattern recognition. Many applications can be found in areas such as manufacturing, medical diagnosis, chemistry and so on. Several object descriptions have been developed. They can be grouped into two categories: region-based and contour-based [35]. Region-based descriptions use both boundary and interior shape information to generate shape descriptors. In this category, a simple shape descriptor has shown its strategic role in shape classification: shape compactness [32, 12, 3 and 17].

Shape analysis starts in the field of psychology where Attneave and Arnoult used shape compactness as a quantitative descriptor [1]. Since then, shape compactness has been used as a shape descriptor for shape analysis. For instance, the medical area uses this feature to assess the spread of tumors [5]. Also, it is used to measure the biocompatibility of new materials which are used for medical devices [25] or for quantitative studies of cell shapes [33], evaluation of mandibular symmetry [11] and so on. Moreover, this feature is crucial in some chemical processes [31 and 2] or geological applications [24].

Although many measures of shape compactness have been proposed [16 and 27], this feature is often associated with the ancestral and well known isoperimetric ratio: $A/4\pi P^2$, where A is the shape area and P is its perimeter [10, 13 and 14]. The isoperimetric ratio gets its minimum value if the shape is a circle. This approach for measuring shape compactness has the following advantages: It is dimensionless, invariant to rotation, translation and scale changes and it is demarcated by the circle. However, Rosenfeld showed aberrant effects when isoperimetric ratio is applied on digital regions. Moreover, he showed that under this approach there are shapes with lower values of compactness than digital disks [29]. This fact has motivated to several research propose technique for measuring shape compactness on digital regions.

In digital space, at least eleven proposals can be found. They can be grouped into three major categories: inner distance, reference shape and geometric pixel properties [26]. Inside of the inner distance approach, we can find the circularity measure, C, by Haralick [15]; the G Factor proposed by Danielsson [8]; a set of circularity measures designed by Di Ruperto and Dempster [9] and the measures of compactness based on border-to-border distances by Wahl [34]. However, Montero and Bribiesca showed that circularity measure, C, has a strong dependence on the sample resolution of the digital region [26]. Meanwhile, Shape Factor G and the measures proposed by Wahl use a distance transform which increases the complexity of this simple feature. Finally, the best measure of DiRuperto and Dempster has several inconsistencies [26].

In the case of the approach based on reference shape we find the Digital Compactness Measure designed by Kim [18], the circularity measure by Bottema [4] and the compactness measure of Peura and Iivarien [28]. In general, these measures compare a digital region with a digital or continuous circle. One problem with this approach is that there are different digital regions that can be representations of a digital circle.

Finally, the geometric pixel property approach has shown to be an adequate method to evaluate shape compactness [3, 5 and 26]. In this category we can find the Discrete Compactness of Bribiesca and the area-perimeter ratios of Bogaert. These measures take the structure of the cells with the digital region is formed. The area-perimeter ratios are compactness measures not invariant to resolution changes; meanwhile Discrete Compactness was designed to be robust at this transformation.

In this paper, we propose a compactness measure for digital regions robust at translations, rotations, and scale transformations. We named it Normalize E-Factor (NEF). By showing that the NEF is robust to scale changes we apply it to shape classification. We also show that the NEF increases classifier efficiency when adding it to the shape description vector.

The rest of this paper is organized as follows. In Section 2, necessary definitions are given. Sections 3 and 4 are devoted to introduce the new shape compactness measure. Experimental results are given in Section 5, while Section 6 provides the conclusions and directions for further research.

2. Boundary of a digital region materials and methods

The perimeter and the area of a shape have shown to be fundamental geometric features in the design of compactness measures [16]. In the digital space, a shape or region is thought as a subset of regular cells under an adjacency relation; this relation defines a connectedness relation between region cells [19]. Traditionally, two connectedness relations are used, 4 and 8-connected. However, the perimeter of a digital region does not have a unique definition whether these relations of connectedness are used.

Rosenfeld describes three different manners for measuring the length of the contour of a digital shape: the sum of the lengths of the crack codes of all pixels on the border of a digital shape, the sum of the areas of the border pixels and whether the digital shape is considered as a set of lattice points, the sum of the edge lengths of the border points [30].

In order to obtain a unique definition of the perimeter length, a digital topology based on a grid cell model was proposed by Kovalevsky [20, 21, 22 and 23]. This topology defines a digital region as a set of discrete elements named k-cells where the basic element is the 0-cell which is a vertex; a pair of 0-cells defines a edge, called 1-cells and the area that forms a set of four 1-cells is named 2-cell where a grid point is the center of a 2-cell [19]. Finally, a set of six 2-cells forms a 3-cell in the three-dimensional digital space. Under this digital topology, the perimeter and the environment area, of a 2D and 3D digital shape respectively, have a unique definition.

In order to define the perimeter of a digital region it is necessary to define the concept of boundary of a region:

Definition 1 (Kovalevsky). *Let S be a digital region formed by a set of 0, 1, 2-cells and let B be a set of 0 and 1-cell subset of S where each element of B is incident both element of S and its complement S'. Then, B is called the boundary of S, $B(S)$.*

Then, the perimeter of S can be stated as:

Definition 2. *The perimeter of S, $P(S)$ is the number of 1-cells in $B(S)$.*

In the case of the area of a digital shape, a common definition is the sum of the unitary area of the pixels of a digital shape. According to Kovalevsky's digital topology, the area of a digital shape S is the number of its 2-cells Bribiesca showed how the area of a digital shape may be represented by a linear expression [6]. This linear expression of the area is known as the Contact Perimeter. Following Kovalevsky's approach we can define the *contact perimeter* as:

Definition 3. Contact perimeter. *Let S be a digital shape, then the contact perimeter, Pc(S), is the difference of the number of 1-cells of S and B(S).*

3. Compactness

Compactness is a concept that tell us how disperse or compact a shape is. More than ten ways have been described to measure this concept [26]. The most referenced expression to obtain a quantitative parameter is the well-known perimeter-area ratio. This ratio was named *classical compactness measure* by Rosenfeld [29], yet it is named circularity measure, thin factor, roundness measure or circularity index [26], There several manners, the most popular is the following:

$$C = \frac{P^2}{A} \quad (1)$$

This way to measure shape compactness can be show to be, in a continuous space, invariant to rotations, scales changes and translations. Moreover, C gives its minimum value when the shape is a circle. However, the measure loss its proprieties when is applied to a digital image. In continuous space the equation (1) is consistent for the shape is built up by an infinity number of points, yet digital regions are formed by a finite number of cells.

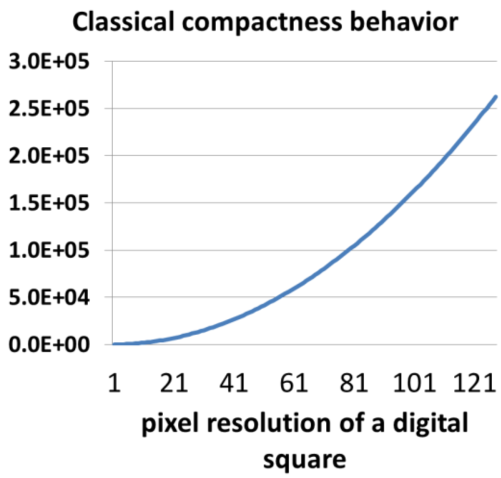

Fig. 1. Exponential effect of the classical compactness measure over digital shapes with different resolutions (color online)

The equation (1) is a ratio of two areas. One of the characteristics with this way to compute shape compactness is that as the resolution of the shape augments, the impact over the computation tends grows exponentially. Fig. 1 shows the resolution effects over digital square shapes at different resolutions.

As mentioned before, several compactness measures for a digital region have been reported. However, only a few of them are invariant to scale transformation. In [26], authors show that the most robust measure is the Normalized Discrete Compactness (NDC) which is also a ratio of areas.

The NDC compares the shape area with the area of the closest square. This comparison by means of a linear expression of areas of both shapes is named *contact perimeter*. The compact perimeter C_D is the number of sides connecting the cells composing the shape. For an example, refer to Fig. 2(a). This has 7 contact sides, drawn in bold. According to [6], the compact perimeter, C_D for a given shape can be obtained by means of the following expression:

$$C_D = \frac{Tn-p}{2} \tag{2}$$

with n representing the number of pixels of the shape, T defining the number of sides of the composing cell ($T = 4$ for a square cell or pixel), and p is the number of external sides of the shape, e.g., the number of sides directly in contact with the shape background. Thus, for the shape shown in Fig. 2(a): $C_D = \frac{Tn-p}{2} = \frac{4\times 7-14}{2} = 7$. Compare with the number of sides drown in bold for the shape depicted in Fig. 2(a).

The NDC uses the contact perimeter, C_D to measure the compactness of a shape. Justo to remember, according to [6], the NDC, C_{DN} of a shape can be computed as:

$$C_{DN} = \frac{C_D - C_{D_{min}}}{C_{D_{max}} - C_{D_{min}}}. \tag{3}$$

with $C_{D_{min}} = n-1$ and $C_{D_{max}} = \frac{Tn-4\sqrt{n}}{2}$ expressing, respectively, the minimum and maximum values that C_{DN} can get.

Equation (2) however exhibits the following phenomenon at the moment of computing the shape compactness. Consider, for the example, the two cases depicted in Fig. 2(b) and 2(c), respectively. As can be seen, the first shape is an elongated one while the second shape is not. Intuitively, the second shape is more compact than the first one, thus a bigger compactness value should be obtained for this shape. However, we can easily see that for both shapes:

$$C_D = \frac{Tn-p}{2} = \frac{4\times 5-12}{2} = 4,$$
$$C_{D_{min}} = n-1 = 5-1 = 4 \text{ and}$$
$$C_{D_{max}} = \frac{Tn-4\sqrt{n}}{2} = \frac{4\times 5-4\sqrt{5}}{2}, \text{ thus:}$$
$$C_{DN} = \frac{C_D - C_{D_{min}}}{C_{D_{max}} - C_{D_{min}}} = \frac{4-4}{\frac{4(5)-4\sqrt{5}}{2}-4} = 0 \tag{4}$$

Both results are correct; however, we can observe that two completely different shapes (one elongated and one not elongated) get the same compactness value of 0.

For elongated shapes as the one shown in Fig. 2(b) according to [6] this would be a desired result; however, the value computed for a shape as shown in Fig. 2(c), in terms of shape classification, would not be a good one.

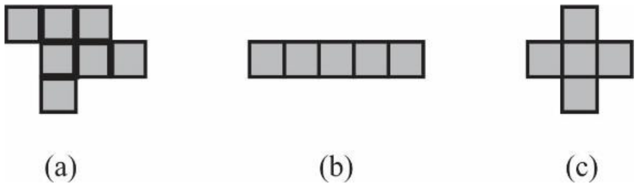

Fig. 2. (a) Contact sides of a shape composed of seven pixels. (b) A digital shape where NDC obtain a compactness value equal to zero

Next, we introduce a compactness measure based on a perimeter ratio which is sensitive to border changes but robust to changes of resolutions on digital shapes.

4. Normalized E-factor

One limitation of many compactness and circularity measures is the resolution of digital region, which limits its applicability in many situations. A common solution to overcome this drawback is to make a comparison between the digital shape under consideration and a reference digital shape [6, 16, 27]. In the case of a digital space, this reference could be a digital square or cube. This comparison must be made with the same number of elements pixels or voxels [6, 7].

In order to make a measure invariant to translations, rotations and scale changes, we decided to use a perimeter-based approach. We measured shape compactness as the ratio of two perimeters: the perimeter of the shape divided by the perimeter of a reference shape, a square shape in the 2-D case and a cube, in the 3-D case. In the 2-D case, the reference shape has as many 2-cells as the shape under study. In the 3-D case, the reference cube has also as many 3-cells as the shape under study.

The proposed ratio is named Normalized E-Factor (NEF) and is given by:

$$NEF = \frac{P_{shape}}{4\sqrt{n}} \tag{5}$$

which is a ratio between the perimeter of the digital shape under consideration and the perimeter of a square with the same number of elements as the original shape. In 3-D digital space, the NEF-3D is the ratio of enclosing surface areas and is given as:

$$NEF-3D = \frac{A_{shape}}{A_{cube}} = \frac{A_{shape}}{3(n-(\sqrt[3]{n})^2)} \tag{6}$$

If the 2-D shape under study has a number of 2-cells, n, which is not 1,4,9,16, …, the reference shape is chosen as the square composed of m=int(n) cells. The same is applicable for the 3-D case.

In order to illustrate the behavior of the NEF we computed for the shapes of Fig. 2. It is clear that these shapes could have different compactness values, however, in the case of the NDC, the set of shapes we got the same compactness value of zero. However, in the case of NEF the object from Fig. 2(a) gets a value of 1.75, while for the objects of Figs. 2(b) and 2(c) we obtain the same value of 1.5.

Besides, the proposed compactness measure satisfies the seven criteria for a good measure of compactness given in [37].

1. Be resolution independent. That is, the threshold used to discriminate circular from non-circular shapes should be the same for any resolution, so that the measure becomes equipment independent.
2. Be consistent. That is, the ordering of objects provided by the measure should be the same at any resolution.
3. Be efficient to calculate, preferably at $O(n)$ complexity.
4. Be defined for all two-dimensional objects.
5. Match the results of human perception.
6. Be a measure of ‘roundness’, rather than ‘non-roundness’.
7. Be resolution independent. That is, the threshold used to discriminate circular from non-circular shapes should be the same for any resolution, so that the measure becomes equipment independent.

5. Experimental results

In this section, we demonstrate the applicability of the new compactness measure by a set of experiments. We first design a database of digital binary images to illustrate that the proposed compactness measure satisfies the criteria established by Ritter [37]. Second, we show how by using a square as a reference shape the *NEF* results to be invariant to geometric transformations. We use a square because in digital space is the most compact shape under any resolution; also, the square shape dominates over the rest of shapes. Third, we show that the *NEF* is a useful descriptor in classification process. For this, we employ the image data base MPEG-7 CE-1.

5.1. Satisfaction of criteria

In this section, we show that the proposed *NEF* measure satisfies the desired criteria defined by Ritter [37], and listed at the end of section 4.

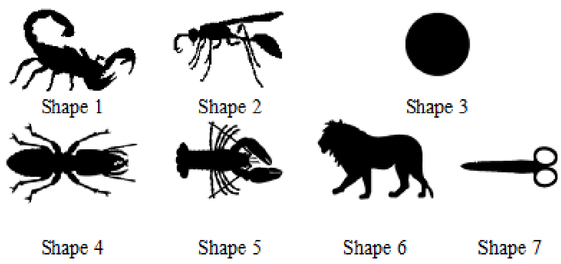

Fig. 3. Set of shapes used to test the proposed compactness measure

In order to show how the *NEF* satisfies criteria number one and two, a set of digital images at different resolutions was obtained to verify its invariance to resolution changes. This set is illustrated in Fig. 3. Figure 4 shows the *NEF* behavior when it is applied over the set of shapes of Fig. 3 at different resolutions, from 64x64 to 248x248 pixels. To study the behavior of the proposed compactness measure under rotation transformations, all shapes at the resolution of 256x256 were rotated from 0 to 185 degrees at steps of 5 degrees each time. Fig. 5 summarizes the results. In both cases, the statistical tests *t* and *z* showed that *NEF* values keep uniform with a significance level of 0.05.

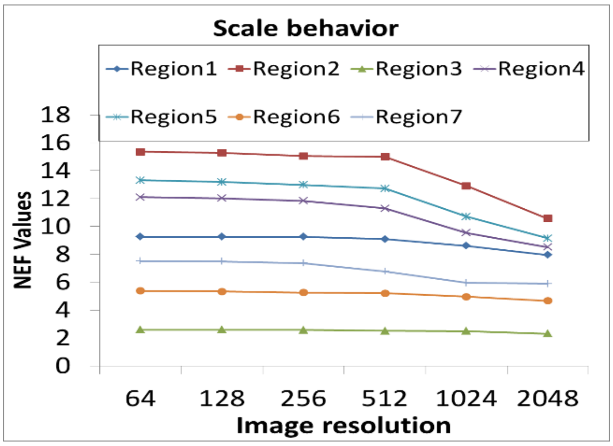

Fig. 4. NEF behavior under scale transformations for the set of shapes of Fig. 3 (color online)

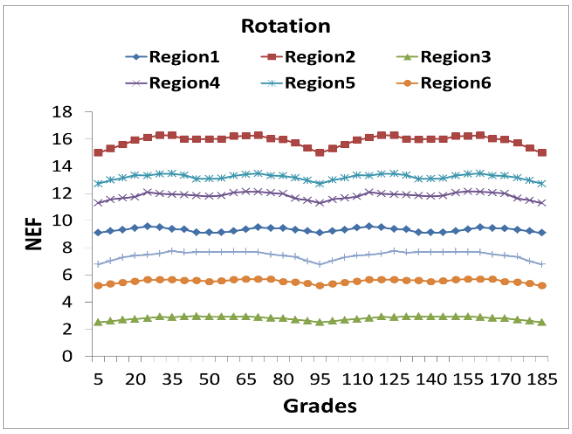

Fig. 5. NEF behavior under rotation transformations for the set of shapes of Fig. 3 (color online)

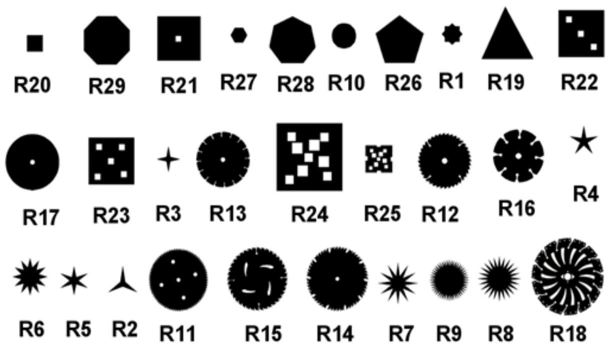

Fig. 6. Shapes ordered according to value and human perception

Third criterion tells us that the measure should be efficient to calculate, preferably at *0(n)* complexity. The *NEF* uses shape perimeters; the computation is carried in one image scan. Therefore, the *NEF* meets this criterion.

Fourth and fifth criteria stipulate that the measure should be defined for all two-dimensional objects and that the results should match those of human perception. In order to show how the *NEF* satisfies these two criteria, a second set of shapes was designed. In this experiment we show how the *NEF* is able to order this second set of shapes according to human perception. The non-ordered set is depicted in Fig. 6(a) while Fig. 6(b) illustrates the same set but ordered according to the *NEF* values. Table 1 shows the *NEF* values in descended order. Thus, the *NEF* satisfies these two criteria.

Criterion six is not applied to *NEF* measure for *NEF* is a compactness measure over a digital space where the most compact shape is the square [38].

The *NEF* was also tested in the 3-D case with six shapes shown in Fig. 7. Table 2 shows their compactness values. As shown in Table 2, intuition is correct, the sphere and the bird were ordered as the most compact shapes, while the octopus and the dragon were ordered as the least compact.

Table 1. NEF values obtained from Fig. 6 shapes ordered from minor to mayor according with the degree of compactness

FEN	Shape	FEN	Shape	FEN	Shape	FEN	Shape
1.000	R20	1.394	R19	2.727	R12	3.423	R14
1.096	R29	1.399	R22	2.748	R16	4.034	R7
1.116	R21	1.548	R17	2.782	R4	4.729	R9
1.129	R27	1.667	R23	3.078	R6	5.534	R8
1.146	R28	2.300	R3	3.107	R5	6.292	R18
1.153	R10	2.409	R13	3.111	R2		
1.216	R26	2.636	R24	3.149	R11		
1.308	R1	2.724	R25	3.187	R15		

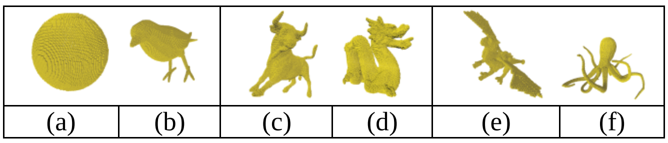

Fig. 7. 3-D digital shapes of (a) a sphere, (b) a bird, (c) a bull, (d) a Chinese dragon, (e) a dragon, (f) an octopus (color online)

Table 2. Compactness values in terms of the   of the 3-D voxelated shapes shown in Fig. 7

Digital region	Area	Volume	NEF
Sphere	30872	276569	1.2171
Bird	18646	71759	1.8372
Bull	64670	357664	2.1764
Chinese dragon	86476	375119	2.8441
Octopus	78312	209910	3.8772
Dragon	58194	114025	4.4187

5.2. Application of the shape classification

We next show that the *NEF* is useful for 2-D binary shape classification. For this, we make use of the MPEG-7 CE-1 data base that contains more than 1000 binary digital shapes belonging to 70 different categories.

We tested the efficiency of the *NEF* combined with the HU moments invariants in the classification process. We evaluated the performance of several classifiers implemented in WEKA testing classifier platform.

For validation accuracy, we used classical statistical 10-crossfolds approach. For the first experiment, we used HU moments alone; results appear reported in Table 3. In order to verify whether the *NEF* is a convenient shape descriptor, we performed a second experiment. In this case, the *NEF* was used together with Hu moments; results appear reported in Table 4.

The accuracy percentages over the MPEG-7 CE-1 data base show that adding the   to the descriptor vector increases efficiency of all the classifiers.

This reinforces the asseveration stated in [12 and 17] that the compactness measure is useful to get better classification results. The NEF keeps this feature without the weakness of NDC measure.

In order to appreciate the discriminatory power of the proposed measure, we proceeded to test it with two non-linearly separable classes. Figs. 8 and 9 depict the two testing classes while Table 5 shows the classification results.

From the results shown in Table 5 we observe that in all cases the addition of the to the describing shape vector helps improving the classifier efficiency. We can also appreciate that the performance of the chosen classifiers is similar. However, the k-NN classifier with provided better classification results.

If we would like to have better efficiencies we would need to 1) add more features to the describing vector, 2) apply a feature selection approach to an initial feature set to get the best combination, or 3) look for a better classifier.

Table 3. Classification results using different classification approaches and only HU moments

Classifier\Descriptor	Hu1	Hu2	Hu3	Hu1,Hu2	Hu1,Hu3	Hu2,Hu3	Hu1,Hu2,Hu3
	Classified						
Naïve_Bayes	17.14%	14.71%	8.64%	27.07%	26.93%	28.43%	37.64%
1NN	27.21%	26.36%	11.21%	48.57%	45.29%	47.14%	58.71%
3NN	23.36%	22.07%	10.07%	41%	38.29%	38.93%	50.86%
5NN	18.29%	21.43%	10.64%	36.57%	32.07%	37.14%	45.71%
j48	20.71%	20.14%	9.86%	42.93%	41.29%	36.00%	51.93%
Bagging	20.36%	22.21%	12.50%	42%	41.29%	42.71%	54.86%
BayesNet	10.50%	11.57%	8.29%	24.07%	24%	25.29%	36.86%
Decision Table	10.50%	11.64%	6.79%	24%	24.29%	25.79%	36.50%
NaiveBayesUpdatable	16.50%	15.07%	8.36%	27.29%	26.64%	28.36%	37.36%
RandomCommittee	26.86%	26.57%	11.21%	48%	44%	44.14%	57.50%
RandomForest	26.21%	26.57%	11.43%	49.07%	47.07%	45.36%	60.43%
RandomSubSpace	17.71%	21%	13.43%	24.50%	23.64%	24.29%	43.93%

Table 4. Classification results using different classification approaches, HU moments and NEF

Classifier\ Descriptor	Hu1 FEN	Hu2 FEN	Hu3 FEN	Hu1,Hu2 FEN	Hu1,Hu3 FEN	Hu2,Hu3 FEN	Hu1,Hu2,Hu3 FEN
	Classified						
Naïve_Bayes	36.29%	38.07%	31%	44.50%	46.86%	49.57%	52.79%
1NN	43.07%	39.71%	32.57%	51.50%	50.43%	51.93%	60.43%
3NN	39.86%	38.50%	29.14%	46.64%	46.93%	47.29%	51.36%
5NN	40.14%	36.64%	29.50%	45.79%	45.21%	46.57%	49.21%
j48	42.50%	43.71%	36.36%	53.86%	55.43%	56.36%	61.43%
Bagging	43.71%	45.36%	40%	55.93%	58.50%	59.14%	64.36%
BayesNet	30%	32.36%	25.21%	40.43%	41.50%	46.93%	51.43%
Decision Table	29.57%	31.14%	25.71%	38.50%	42.29%	44.14%	44.43%
Naïve_Bayes Updatable	36.29%	37.86%	30.79%	44.79%	48.79%	50.57%	53%
Random Committee	44.14%	44.21%	35.29%	57.50%	58%	58.43%	68.07%
Random Forest	44.93%	46.79%	38%	60.36%	62.14%	62.21%	69.71%
Random Sub Space	24.36%	25.36%	21.93%	46.86%	45.50%	46.57%	55.93%

The accuracy percentages over the MPEG-7 CE-1 data base show that adding the *NEF* to the descriptor vector increases efficiency of all the classifiers. This reinforces the asseveration stated in [12 and 17] that the compactness measure is useful to get better classification results. The *NEF* keeps this feature without the weakness of NDC measure.

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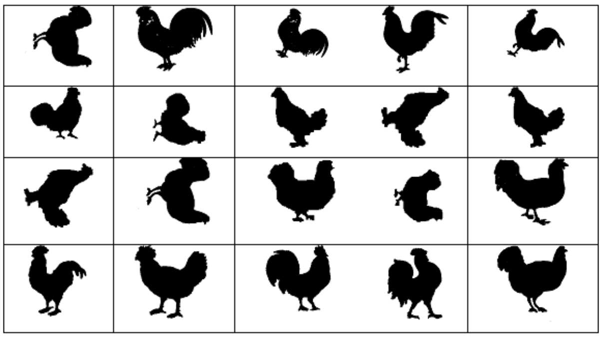

Fig. 8. Set of different chicken shapes for the first class

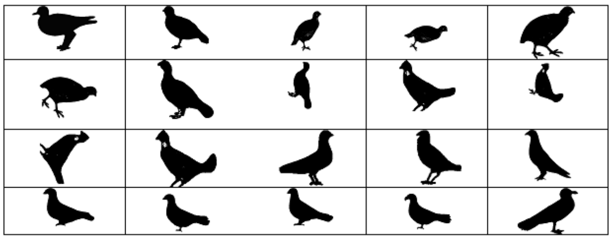

Fig. 9. Set of different bird shapes for the second class

Table 5. (a) Classification accuracy using 10 cross fold validation and 1-NN over the classes for Figs. 8 and 9 (b) Classification accuracy using 10 cross fold validation and j48 tree over the classes in Figs. 8 and 9. (c) Classification accuracy using 10 cross fold validation and NaiveBayese over the classes for Figs. 8 and 9

1-NN with 10 crossfold		Tree (j48) with 10 crossfold		NaiveBayes with 10 crossfold	
Test 1. First Hu moment					
Correctly classified	67%	Correctly classified	55%	Correctly classified	55%
misclassified	37%	misclassified	45%	misclassified	45%
Test 2. First Hu moment and NEF					
Correctly classified	83%	Correctly classified	75%	Correctly classified	75%
misclassified	17%	Misclassified	25%	misclassified	25%
Test 3. First and second Hu moments					
Correctly classified	72%	Correctly classified	50%	Correctly classified	72.5%
misclassified	28%	misclassified	50%	misclassified	27.5%
Test 4. First and second Hu moments and NEF					
Correctly classified	93%	Correctly classified	75%	Correctly classified	80%
Misclassified	7%	Misclassified	25%	misclassified	20%

6. Conclusions

In this work, a compactness measure for a digital region composed by square cells was proposed. Nowadays, every measure of compactness is a function where many domain elements could have one element of codomain. In others words, many shapes could have a same compactness value. Therefore, the NEF being a compactness measure is possible that many shapes could have the same value of NEF.

Through experimentation, we have seen that the proposed measure describe well the morphological changes that the digital region has. Obtained results have are well correlated with human perception. The NEF is a reference shape measure where the value of compactness is determined by the ratio between two perimeters, the perimeter of the digital region and the perimeter of the square with similar area as reference shape. The NEF can support the changes of resolution on the digital region and it can be applied for 3-D shapes as well. Through experimentation we have seen that the NEF is simple and may be a useful descriptor for classifying digital regions. Moreover, the NEF satisfies five of the six criteria established by Ritter defining a “good” compactness measure.

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*Corresponding author: david.gutierrez@leon.tecnm.mx