Optical soliton perturbation with quadratic-cubic nonlinearity by the method of undetermined coefficients

MIR ASMA[a], W. A. M. OTHMAN[a] , B. R. WONG[a], ANJAN BISWAS[b,c*]
[a]*Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia*
[b]*Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria -0008, South Africa*
[c]*Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, PO Box-80203, Jeddah-21589, Saudi Arabia*

This paper obtains optical soliton solution to perturbed nonlinear Schrödinger's equation with quadratic-cubic nonlinearity with the usage of undetermined coefficients. Bright, dark, singular and W-shaped solitons are retrieved with this scheme. The existence criteria for these solitons are also given.

1. Introduction

The dynamics of optical soliton molecules in optical fibers has been extensively studied during the past few decades [1-10]. There are several forms of nonlinear media where these solitons are studied. A recently reported nonlinear medium is the quadratic-cubic (QC) type, that first appeared in 2011, where variational approach has laid down the parameter dynamics of the soliton [6]. Since its first appearance, few results have been reported with regards to QC nonlinearity. These include conservation laws, exact and analytical soliton solutions that are obtained by the application of traveling waves, semi-inverse variational principle, direct integration method and others [2-4, 8]. This paper will be addressing the perturbed nonlinear Schrödinger's equation (NLSE) (the governing equation), with QC nonlinearity where perturbation terms are all of Hamiltonian type. This does not destroy the integrability aspect of the perturbed version of NLSE. The method of undetermined coefficients will be implemented to extract these soliton solutions that are also supported by numerical schemes. The details appear in the following sections.

1.1. Governing Model

The governing resonant NLSE with perturbation terms that is studied in nonlinear optics is given in its dimensionless form as [2-4, 9, 10]:

$$iq_t + aq_{xx} + \left(b_1|q| + b_2|q|^2\right)q = i\left[\alpha q_x + \lambda\left(|q|^2 q\right)_x + \theta\left(|q|^2\right)_x q\right] \tag{1}$$

In (1), $q(x,t)$ represents the complex-valued wave profile with two independent variables x and t which represents spatial and temporal components respectively. On the left side of equation (1), the first term is the linear temporal evolution, while from the second term, a is the coeffficient of group velocity dispersion (GVD). The two nonlinear terms are with b_1 and b_2 that are quadratic and cubic nonlinear terms respectively. On the right side of (1) are the perturbation terms. The coefficient of α is inter-modal dispersion, while the coefficients of λ and θ respectively are self-steepening term and nonlinear dispersion. All of these parameters a, b_j for $j = 1, 2, \alpha, \lambda$ and θ are real-valued constants. The three terms on the right side of (1) constitute Hamiltonian perturbation terms. Equation (1) is therefore rendered integrable, since it passes the Painleve test of integrability. The following section will detail the derivation of soliton solutions to (1).

2. Soliton Solutions

In order to get started the following assumption is made towards the soliton solution structure to (1) [2-4]:

$$q(x,t) = P(x,t)e^{i\phi(x,t)} \tag{2}$$

where $P(x,t)$ is the amplitude portion of the soliton. The phase component is

$$\phi(x,t) = -\kappa x + \omega t + \theta_0. \tag{3}$$

where, κ represents the soliton frequency, ω is the wave number and θ_0 is the phase constant. Substituting (2) into (1) and equating real and imaginary parts leads to [2, 13]

$$\left(\omega+\alpha\kappa+a\kappa^2+b_1\right)P+\left(b_2-\lambda\kappa\right)P^3+a\frac{\partial^2 P}{\partial x^2}=0,$$

(4)

and

$$\left(v+2a\kappa+\alpha\right)+\left(3\lambda+2\theta\right)P^2=0.$$ (5)

The imaginary part equation yields, after setting the coefficients of the linearly independent functions to zero,

$$v=-2a\kappa-\alpha$$ (6)

and

$$3\lambda+2\theta=0.$$ (7)

This gives the speed of the soliton irrespective of the type of soliton searched. Also, (7) gives the constraint relation between the perturbation parameters. Equation (4) will now be analyzed in the next few subsections to locate bright, dark and singular solitons.

2.1. Bright Solitons

In order to obtain bright soliton solutions, the starting assumption is [2-4]

$$P(x,t)=\frac{A}{\left(D+\cosh\tau\right)^p},$$ (8)

where

$$\tau=B(x-vt)$$ (9)

and A is the amplitude of the soliton, B is its inverse width while D is an additional parameter that is related to A and B and this connection will be revealed shortly. The value of the unknown parameter p will also be disclosed from balancing principle applied to the equation for $P(x,t)$. Substituting (8) into (4) gives:

$$\frac{p(2p+1)aB^2D}{\left(D+\cosh\tau\right)^{1+p}}-\frac{b_1A}{\left(D+\cosh\tau\right)^{2p}}+\frac{\left(\kappa\lambda-b_2\right)A^2}{\left(D+\cosh\tau\right)^{3p}}+$$

$$\frac{\omega+\alpha\kappa-a\left(B^2p^2-\kappa^2\right)}{\left(D+\cosh\tau\right)^p}-\frac{ap(p+1)\left(D^2-1\right)B^2}{\left(D+\cosh\tau\right)^{2+p}}=0.$$

(10)

By the aid of balancing principle, equating the exponents $3p$ and $p+2$ gives

$$p=1.$$ (11)

Subsequently, setting the coefficients of the linearly independent functions in (10) to zero yields:

$$\omega=a\left(B^2-\kappa^2\right)-\alpha\kappa,$$ (12)

$$A=\frac{3\sqrt{2}aB^2}{\sqrt{9aB^2\left(b_2-\kappa\lambda\right)+2b_1^2}},$$ (13)

and

$$D=\frac{b_1\sqrt{2}}{\sqrt{9aB^2\left(b_2-\kappa\lambda\right)+2b_1^2}}.$$ (14)

The pair of relations (13) and (14) implies the restriction:

$$9aB^2\left(b_2-\kappa\lambda\right)+2b_1^2>0$$ (15)

for existence of these bright solitons. Hence, the bright 1-soliton solution to (1) is:

$$q(x,t)=\frac{A}{D+\cosh\left[B(x-vt)\right]}e^{i\left(-\kappa x+\omega t+\theta_0\right)},$$ (16)

where the parameter definitions and constraints are indicated.

The following Fig.1 is the profile of a bright 1-soliton solution of the perturbed NLSE given by (1). Here the parameter values are

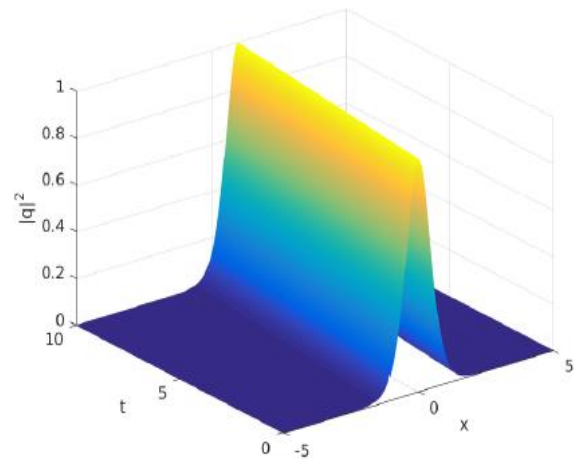$$a=0.5,\ b_1=2.0,\ b_2=0.25,\ \alpha=\lambda=1.0,\ \theta=-1.5.$$

Fig. 1. Profile of a bright soliton

2.2. Dark Solitons

In this case the starting hypothesis is:

$$P(x,t)=\left(A+B\tanh\tau\right)^p,$$ (17)

where A and B are free parameters and

$$\tau = \mu(x - v t), \qquad (18)$$

with p being the unknown parameter that needs to be determined by the aid of balancing principle. Upon substituting (17) into (4), the real part simplifies to:

$$4(2p+1)a\,p\,\mu^2 A\left(A+B\ \tanh\tau\right)^{p+1} -$$
$$2a\,\mu^2 p\left(1+p\right)\left(A+B\ \tanh\tau\right)^{p+2} -$$
$$2a\,\mu^2 p\left(p-1\right)\left(A^2-B^2\right)^2\left(A+B\ \tanh\tau\right)^{p-2} -$$
$$4b_1 B^2\left(A+B\ \tanh\tau\right)^{2p} -$$
$$2B^2\left(b_2-\lambda\kappa\right)\left(A+B\ \tanh\tau\right)^{3p} -$$
$$4p^2 a\,\mu^2\left(3A^2-B^2\right) +$$
$$2B^2\left(\omega+\alpha\kappa+a\kappa^2\right)\left(A+B\ \tanh\tau\right)^{p} +$$
$$4\mu^2 a A\,p\left(2p+1\right)\left(A^2-B^2\right)\left(A+B\ \tanh\tau\right)^{p-1} = 0. \qquad (19)$$

Similarly, as in bright solitons, balancing principle yields the same value of p as in (11). Next, setting the coefficients of linearly independent functions to zero leads to the following relations with the soliton parameters:

$$A = \pm B = \pm\frac{b_1}{3\left(\kappa\lambda-b_2\right)}, \qquad (20)$$

$$\mu = \pm\frac{1}{6}\frac{b_1\sqrt{2}}{\sqrt{a\left(\kappa\lambda-b_2\right)}} \qquad (21)$$

$$\omega = -a\kappa^2 + 4a\mu^2 - \alpha\kappa \qquad (22)$$

Thus, relation (21) introduces the constraint:

$$a\left(\kappa\lambda-b_2\right) > 0. \qquad (23)$$

Therefore, dark 1-soliton solution is given by:

$$q(x,t) = A\left\{1\pm\tanh\left[\mu\left(x-vt\right)\right]\right\}e^{i\left(-\kappa x+\alpha t+\theta_0\right)}, \qquad (24)$$

with the parameter definitions and restrictions in place.

The following Figure-2 is the profile of a dark 1-soliton solution of the perturbed NLSE given by (1). Here the parameter values are

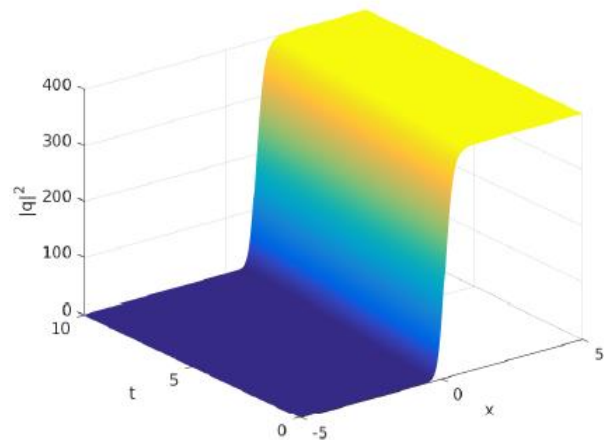$a = 0.5$, $b_1 = 3.0$, $b_2 = -1.2$, $\alpha = \lambda = 1.0$ and $\theta = -1.5$.

Fig. 2. Profile of a dark soliton

2.3. Singular Solitons (Type-I)

For singular solitons of the first kind the assumption is [2-4]:

$$P(x,t) = \frac{A}{\left(D+\sinh\tau\right)^p}, \qquad (25)$$

and in this case A, B and D are all free parameters with the unknown index p. Substituting (25) into (4) leads to

$$\frac{-B^2\left(2p+1\right)aDp}{E^{1+p}} + \frac{Ab_1}{E^{2p}} + \frac{A^2\left(-\kappa\lambda+b_2\right)}{E^{3p}} +$$
$$\frac{\left(B^2 ap^2-a\kappa^2-\alpha\kappa-\omega\right)}{E^p} + \frac{B^2 ap\left(D^2+1\right)\left(1+p\right)}{E^{2+p}}, \qquad (26)$$

Once again, $p = 1$ as given by (11). Similarly, the coefficients of the linearly independent functions lead to the same expression for wave number as in (12). However, the free parameters are related as

$$A = \pm\frac{3\sqrt{2}aB^2}{\sqrt{9B^2 a\left(\kappa\lambda-b_2\right)-2b_1^2}} \qquad (27)$$

and

$$D = \frac{b_1}{3aB^2}A \qquad (28)$$

$$\omega = B^2 a - a\kappa^2 - \alpha\kappa$$

The pair (27) and (28) introduces the restriction:

$$9B^2a\left(\kappa\lambda - b_2\right) - 2b_1^2 > 0. \qquad (29)$$

for existence of these singular solitons. Hence, the singular 1-soliton solution to (1) is:

$$q(x,t) = \frac{A}{D + \sinh\left[B\left(x - vt\right)\right]}e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \quad (30)$$

for designated parameters.

2.4. Singular Solitons (Type-II)

For the second type of singular solitons, the starting asssumption is:

$$P(x,t) = \left(A + B\coth\tau\right)^p, \qquad (31)$$

where A and B are free parameters and τ carries the same definition as in (18). Equation (4), by virtue of (31), reduces to:

$$4\left(2p+1\right)ap\mu^2 A\left(A + B\coth\tau\right)^{p+1} -$$
$$2a\mu^2 p\left(1+p\right)\left(A + B\coth\tau\right)^{p+2} -$$
$$2a\mu^2 p\left(p-1\right)\left(A^2 - B^2\right)^2\left(A + B\coth\tau\right)^{p-2} -$$
$$4b_1 B^2\left(A + B\coth\tau\right)^{2p} -$$
$$2B^2\left(b_2 - \lambda\kappa\right)\left(A + B\coth\tau\right)^{3p} -$$
$$4p^2 a\mu^2\left(3A^2 - B^2\right) +$$
$$2B^2\left(\omega + \alpha\kappa + a\kappa^2\right)\left(A + B\coth\tau\right)^p +$$
$$4\mu^2 aAp\left(2p+1\right)\left(A^2 - B^2\right)\left(A + B\coth\tau\right)^{p-1} = 0. \qquad (32)$$

The unknown exponent p turns out to be the same as in (11). Finally, as in dark solitons, the parameters A, B, μ and ω receive the same values as seen in (20) - (22) together with the same restriction as given by (23). Thus, finally, the second form of singular solitons is written as:

$$q(x,t) = A\left\{1 \pm \coth\left[\mu\left(x - vt\right)\right]\right\}e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \qquad (33)$$

for the defined parameters.

2.5. W- Shaped Soliton

Here, the starting assumption is [9]:

$$P(x,t) = \beta + \rho\operatorname{sech}\tau \qquad (34)$$

Substituting $P(x,t)$ from (34) into (4) gives

$$\left\{2a\mu^2 + p^2\left(\kappa\lambda - b_2\right)\right\}\left(\beta + \rho\operatorname{sech}\tau\right)^3 -$$
$$\left(6a\beta\mu^2 + b_1\rho^2\right)\left(\beta + \rho\operatorname{sech}\tau\right)^2 +$$
$$\left\{\mu^2 a\left(6\beta^2 - \rho^2\right) + \rho^2\left(\omega + \alpha\kappa + a\kappa^2\right)\right\}$$
$$\left(\beta + \rho\operatorname{sech}\tau\right) - \beta\mu^2\left(2\beta^2 - \rho^2\right)a = 0. \qquad (35)$$

Balancing principle again yields $p = 1$. The coefficients of linearly independent functions give the parameters as:

$$\beta = -\frac{b_1}{3\kappa\lambda - b_2}, \qquad (36)$$

$$\omega = -\frac{9\kappa\left(-\kappa\lambda + b_2\right)\left(a\kappa + \alpha\right) + 2b_1^2}{9\left(\kappa\lambda - b_2\right)} \qquad (37)$$

$$\mu = \pm\frac{b_1}{3\sqrt{a\left(\kappa\lambda - b_2\right)}}, \qquad (38)$$

$$\rho = \pm\frac{b_1}{3\left(-\kappa\lambda + b_2\right)}. \qquad (39)$$

These parameters pose the same constraint as (23). Thus, the $W-$ shaped soliton is finally written as:

$$q(x,t) = \left\{\beta + \rho\operatorname{sech}\left[B(x - vt)\right]\right\}e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \qquad (40)$$

where all parameters and constraints are clearly presented.

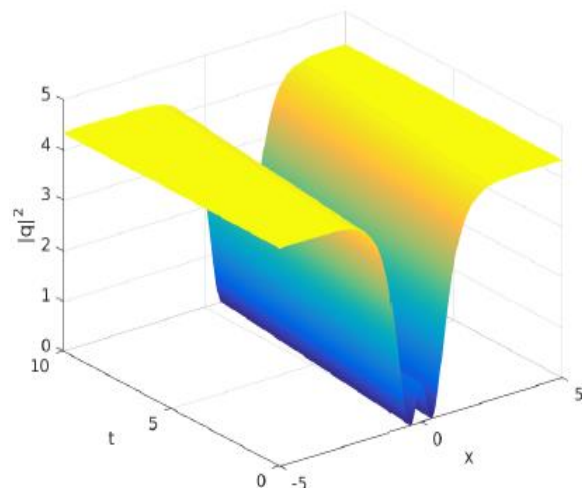$a = 0.25,\ b_1 = 2.0,\ b_2 = -2.3,\ \alpha = 1.0,\ \lambda = 0.9$ and $\theta = -1.35$.

Fig. 3. Profile of a W-shaped soliton

3. Conclusions

This paper secures bright, dark, singular and $W-$ shaped soliton solutions to the perturbed NLSE with QC nonlinearity. The perturbation terms are all of Hamiltonian type. The existence criteria of these solitons are also given. The numerical simulation of these solitons support the analytical results. The method of undetermined coefficients was implemented to retrieve these soliton solutions from the model. This paper is a sequel to an earlier reported result where optical soliton solutions were obtained for the same model by traveling wave hypothesis [3]. The advantage of this scheme is that one can extract dark and $W-$ shaped solitons which are not possible by traveling wave hypothesis. On the other hand, combo-soliton solutions are possible with traveling wave hypothesis which the current method fails to obtain. In conclusion, it is imperative to study any soliton model by the aid of as many integration schemes as possible. It is only then a complete spectrum of soliton solutions, along with necessary constraints, is recovered.

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*Corresponding author: biswas.anjan@gmail.com