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Optical solitons in dual-core fibers with G'/G-expansion scheme

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This paper obtains dark and singular 1-soliton solutions in dual-core fibers by the aid of G'/G-expansion scheme. The constraint conditions, for the existence of the soliton solutions, are listed. Additionally, a couple of other solutions known as singular periodic solutions, fall out as a by-product of this scheme. This scheme however fails to retrieve bright soliton solutions.

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1. Introduction

Optical solitons is one of the most fascinating areas of research at the present time. These soliton molecules are basic ingredients for information transfer, through optical fibers for trans-continental and trans-oceanic distances [1-30]. Therefore, it is imperative to address the dynamics of these soliton pulses from a mathematical perspective. This will lead to a deeper understanding of the engineering aspects of these solitons.

This paper will study these solitons in dual-core optical fibers from a purely mathematical standpoint. The focus of this paper therefore will be to extract exact 1-soliton solution for the governing model. This model is described the coupled nonlinear Schrödinger's equation (NLSE). There are several integration tools available to solve the model. A few of them are traveling waves, homotopy analysis method, variational principle, Kudryashovs method, simplest equation method, tanh-expansion scheme, extended tanh method and several others. This paper will however address one such integration mechanism. This is the G'/G-expansion scheme that will retrieve dark and singular soliton solutions for the coupled NLSE. Additionally, singular soliton solutions will naturally emerge as a by-product of this integration tool.

2. The model

Pulse propagation in a decoupled two-core fibers has distinction from continuous wave propagation. In a conventional two core fiber, pulse propagation has been studied extensively by solving the coupled mode equations; where the light coupling between the two cores is characterized by a structure dependent parameter called the coupling coefficients. The model for decoupled NLSE read as [10, 15]:

$$i\left(\frac{\partial \psi_1}{\partial x} + \alpha_1 \frac{\partial \psi_2}{\partial t}\right) + \alpha_2 \frac{\partial^2 \psi_1}{\partial t^2} + \alpha_3 \left|\psi_1^2\right| \psi_1 + \alpha_4 \psi_2 = 0 \tag{1}$$

$$i\left(\frac{\partial \psi_2}{\partial x} + \alpha_1 \frac{\partial \psi_1}{\partial t}\right) + \alpha_2 \frac{\partial^2 \psi_2}{\partial t^2} + \alpha_3 \left|\psi_2^2\right| \psi_2 + \alpha_4 \psi_1 = 0 \tag{2}$$

where ψ_1 and ψ_2 are the field envelopes, while x is the propagation co-ordinate and $1/\alpha_1$, α_2 and α_4 are group velocity mismatch, group velocity dispersion and linear coupling coefficient, respectively. It may also be noted that α_3 is defined by $\alpha_3 = 2\pi n_2 / \vartheta A_{eff}$, where n_2 , ϑ and A_{eff} are nonlinear refractive index, the wavelength and effective mode area of each wavelength, respectively. These details are already known [1, 2, 26].

The rest of the article is organized as follows, in Section-3 the G'/G-expansion scheme has been discussed.

The two kinds of soliton solutions to the application: decoupled NLSEs for two-core fiber are constructed in next Section 4. In last Section 5, the conclusion was drawn.

3. Overview of G'/G-expansion scheme

In this section, the G'/G-expansion scheme [9, 12] has been discussed to find traveling wave solutions of nonlinear evolution equations and then subsequently it will be applied to solve the coupled NLSE, in next section.

We assume that the given nonlinear evolution equation for $u(x,t)$ takes the form

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, ..., \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, ...\right) = 0 \quad (3)$$

where u is an unknown function and P is a polynomial of u and its partial fractional derivatives along with the involvement of higher order derivatives and nonlinear terms. To find the optical soliton, the G'/G-expansion scheme can be performed by using the following steps:

Step-1: To find the travelling wave solutions of equation (3), the transformation wave variable is being used. After substituting transformation wave variable into equation (1), the following ODE can be obtained

$$Q(U, U', U'', ...) = 0 \quad (4)$$

Step-2: The equation (4) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

Step-3: Introduce the solution $U(\xi)$ of equation (4) in the finite series form

$$U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i \quad (5)$$

where a_i are real constants with $a_m \neq 0$ and while m is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ODE.

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \quad (6)$$

Step-4: Determining m can be accomplished by balancing the linear term of highest order derivatives with the highest order nonlinear term in equation (3).

Step-5: Substituting the general solution of equation (6) together with equation (5) into equation (4), which yields a system of algebraic equation involving powers of G'/G. Equating the coefficients of each power of G'/G to zero, that gives a system of algebraic equations for a_i, λ and μ. Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. It may be noted that the general solution of equation (6) is

$$G(\xi) = Ae^{\left(-\lambda + \sqrt{\lambda^2 - 4\mu}\right)\xi/2} + Be^{\left(-\lambda - \sqrt{\lambda^2 - 4\mu}\right)\xi/2} \quad (7)$$

The derivative of $G(\xi)$ can be obtained from above equation in the following form:

$$G'(\xi) = -\frac{\lambda}{2} e^{(-\lambda/2)\xi} \left[\begin{gathered} (A+B)\cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) \\ +(A-B)\sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) \end{gathered} \right] + \frac{\sqrt{\lambda^2 - 4\mu}}{2} e^{(-\lambda/2)\xi} \left[\begin{gathered} (A+B)\sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) \\ +(A-B)\cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) \end{gathered} \right] \quad (8)$$

In the form of ratio, $G'(\xi)$ and $G(\xi)$ are expressed as:

$$\frac{G'(\xi)}{G(\xi)} = \left(\frac{-\lambda + \sqrt{\lambda^2 - 4\mu}}{2}\right) \times \frac{(A+B)\sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + (A-B)\cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{(A+B)\cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + (A-B)\sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)} \quad (9)$$

In the following section, the decoupled NLSEs have been considered to construct the different types of soliton solutions.

4. Application to dual-core optical fibers

Let us demonstrate the application of the G'/G-expansion scheme for finding the new two types of exact traveling wave solutions of decoupled NLSE for two-core fiber. For this, we convert equations (1) and (2) into nonlinear ordinary differential equations using the following wave variable transformation:

$$\psi_1(x.t)=U(\xi)e^{i\eta},\ \psi_2(x.t)=V(\xi)e^{i\eta} \quad (10\text{-}1)$$

and

$$\psi_1^*(x.t)=U(\xi)e^{-i\eta},\ \psi_2^*(x.t)=V(\xi)e^{-i\eta} \quad (10\text{-}2)$$

where $\xi=kx-\omega t$ and $\eta=lx-vt$. k , ω , l and v are unknown constants that are to be determined.

Thus, from the wave transformation, it follows

$$\begin{aligned}&\alpha_2\omega^2U''+i(k-\alpha_2\omega(\omega+v))U'\\&-i\alpha_1\omega V'+\alpha_3U^3\\&-(l+\alpha_2\omega v)U+\alpha_4V=0\end{aligned} \quad (11)$$

$$\begin{aligned}&\alpha_2\omega^2V''+i(k-\alpha_2\omega(\omega+v))V'-i\alpha_1\omega U'\\&+\alpha_3V^3-(l+\alpha_2\omega v)V+\alpha_4U=0\end{aligned} \quad (12)$$

Balancing U'' with U^3 in equation (11) and V'' with V^3 in equation (12), using homogenous balance method, gives $m+2=3m\Rightarrow m=1$ and $n+2=3n\Rightarrow n=1$, respectively.

It must be noted that equation (5) has the the following forms:

$$U(\xi)=A_1\left(\frac{G'}{G}\right)+A_0,\ V(\xi)=B_1\left(\frac{G'}{G}\right)+B_0 \quad (13)$$

where A_0 , A_1 , B_0 and B_1 are constants to be determined later. From equation (13), we obtain

$$U'(\xi)=-A_1\left(\frac{G'}{G}\right)^2-\lambda A_1\left(\frac{G'}{G}\right)^1-\mu A_1 \quad (14)$$

$$\begin{aligned}U''(\xi)=&\,2A_1\left(\frac{G'}{G}\right)^3+3\lambda A_1\left(\frac{G'}{G}\right)^2\\&+(\lambda^2+2\mu)A_1\left(\frac{G'}{G}\right)^1+\mu\lambda A_1\end{aligned} \quad (15)$$

$$\begin{aligned}U^3(\xi)=&\,A_1^3\left(\frac{G'}{G}\right)^3+3A_0A_1^3\left(\frac{G'}{G}\right)^2\\&+3A_0^2A_1\left(\frac{G'}{G}\right)+A_0^3\end{aligned} \quad (16)$$

$$V'(\xi)=-B_1\left(\frac{G'}{G}\right)^2-\lambda B_1\left(\frac{G'}{G}\right)^1-\mu B_1 \quad (17)$$

$$\begin{aligned}V''(\xi)=&\,2B_1\left(\frac{G'}{G}\right)^3+3\lambda B_1\left(\frac{G'}{G}\right)^2\\&+(\lambda^2+2\mu)B_1\left(\frac{G'}{G}\right)^1+\mu\lambda B_1\end{aligned} \quad (18)$$

$$\begin{aligned}V^3(\xi)=&\,B_1^3\left(\frac{G'}{G}\right)^3+3B_0B_1^3\left(\frac{G'}{G}\right)^2\\&+3B_0^2B_1\left(\frac{G'}{G}\right)+B_0^3\end{aligned} \quad (19)$$

After substituting equations (14)-(19) into equations (11) and (12) and collecting all terms with the same order of G'/G together, we convert the left-hand side of equations (11) and (12) into a polynomial in G'/G,

$$2\alpha_2\omega^2A_1+\alpha_3A_1^3=0 \quad (20)$$

$$2\alpha_2\omega^2B_1+\alpha_3B_1^3=0 \quad (21)$$

$$\begin{aligned}&3\alpha_2\omega^2\lambda A_1-i\big(k-\alpha_2\omega(\omega+v)\big)A_1\\&+i\alpha_1\omega B_1+3\alpha_3A_0A_1^2=0\end{aligned} \quad (22)$$

$$\begin{aligned}&3\alpha_2\omega^2\lambda B_1-i\big(k-\alpha_2\omega(\omega+v)\big)B_1\\&+i\alpha_1\omega A_1+3\alpha_3B_0B_1^2=0\end{aligned} \quad (23)$$

$$\begin{aligned}&\alpha_2\omega^2A_1(\lambda^2+2\mu)-i\lambda\big(k-\alpha_2\omega(\omega+v)\big)A_1\\&+i\alpha_1\omega\lambda B_1+3\alpha_3A_1A_0^2\\&-(l+\alpha_2\omega v)A_1+\alpha_4B_1=0\end{aligned} \quad (24)$$

$$\begin{aligned}&\alpha_2\omega^2B_1(\lambda^2+2\mu)-i\lambda\big(k-\alpha_2\omega(\omega+v)\big)B_1\\&+i\alpha_1\omega\lambda A_1+3\alpha_3B_1B_0^2\\&-(l+\alpha_2\omega v)B_1+\alpha_4A_1=0\end{aligned} \quad (25)$$

$$\begin{aligned}&\alpha_2\omega^2A_1\lambda\mu-i\mu\big(k-\alpha_2\omega(\omega+v)\big)A_1\\&+i\alpha_1\omega\mu B_1+\alpha_3A_0^3\\&-(l+\alpha_2\omega v)A_0+\alpha_4B_0=0\end{aligned} \quad (26)$$

$$\begin{aligned}&\alpha_2\omega^2B_1\lambda\mu-i\mu\big(k-\alpha_2\omega(\omega+v)\big)B_1\\&+i\alpha_1\omega\mu A_1+\alpha_3B_0^3\\&-(l+\alpha_2\omega v)B_0+\alpha_4A_0=0\end{aligned} \quad (27)$$

From equations (20) and (21), we can obtain $A_1 = B_1 = \omega\sqrt{-2\alpha_2/\alpha_3}$, which imposes $\alpha_2\alpha_3 < 0$ Let us choose that when $\lambda = 0$ and $A_0 = 0$, the first type exact traveling wave solutions are obtained as:

$$\psi_{1_1} = \psi_{2_1} = -\omega\sqrt{\frac{2\alpha_2\mu}{\alpha_3}} \times \frac{(A+B)\sinh(\sqrt{-\mu}\xi) + (A-B)\cosh(\sqrt{-\mu}\xi)}{(A+B)\cosh(\sqrt{-\mu}\xi) + (A-B)\sinh(\sqrt{-\mu}\xi)} \times \exp[i(lx - vt)] \tag{28}$$

The dark 1-soliton solution is obtained, when $\mu < 0$ and $A = B = 1$ was considered

$$\psi_{1_{11}} = \psi_{2_{11}} = -\omega\sqrt{\frac{2\alpha_2\mu}{\alpha_3}}\tanh(\sqrt{-\mu}\xi) \times \exp[i(lx - vt)] \tag{29}$$

The singular 1-soliton solution is obtained, when $\mu < 0$ and $A = -B = 1$ was considered.

$$\psi_{1_{12}} = \psi_{2_{12}} = -\omega\sqrt{\frac{2\alpha_2\mu}{\alpha_3}}\coth(\sqrt{-\mu}\xi) \times \exp[i(lx - vt)] \tag{30}$$

The singular periodic solution is obtained, when $\mu \geq 0$ and $A = B = 1$ was considered.

$$\psi_{1_{13}} = \psi_{2_{13}} = \omega\sqrt{\frac{2\alpha_2\mu}{\alpha_3}}\tan(\sqrt{\mu}\xi) \times \exp[i(lx - vt)] \tag{31}$$

Another singular periodic solution is recovered, when $\mu \geq 0$ and $A = -B = 1$ was considered.

$$\psi_{1_{14}} = \psi_{2_{14}} = -\omega\sqrt{\frac{2\alpha_2\mu}{\alpha_3}}\cot(\sqrt{\mu}\xi) \times \exp[i(lx - vt)] \tag{32}$$

After solving the above system of equations, we have the following different cases:

Case-I:

$$A_1 = -B_1 = \omega\sqrt{-\frac{2\alpha_2}{\alpha_3}},\ A_0 = -B_0 = \omega\frac{\lambda}{2}\sqrt{-\frac{2\alpha_2}{\alpha_3}}$$

$$k = \frac{\omega}{2}(\alpha_2\omega + \alpha_2 v + 2\alpha_1),$$

$$l = -\alpha_2\omega^2 A_0^2 + \frac{2\alpha_2\omega^2\mu - \alpha_2\omega v + 2\alpha_4}{2}$$

Case-II:

$$A_1 = -B_1 = \omega\sqrt{-\frac{2\alpha_2}{\alpha_3}},\ A_0 = -B_0 = \omega\frac{\lambda}{2}\sqrt{-\frac{2\alpha_2}{\alpha_3}}$$

$$k = \frac{\omega}{2}(\alpha_2\omega + \alpha_2 v - 2\alpha_1),$$

$$l = -\alpha_2\omega^2 A_0^2 + \frac{2\alpha_2\omega^2\mu - \alpha_2\omega v - 2\alpha_4}{2}$$

where k, l, v and ω are free parameters. Thus, the second type exact travelling wave solutions to decoupled NLSEs are obtained as:

$$\psi_{1_2} = \left(\omega\sqrt{-\frac{\alpha_2}{2\alpha_3}}\frac{-\lambda + \sqrt{\lambda^2 - 4\mu}}{2}\right) \times \exp[i(lx - vt)] \tag{33}$$

$$\frac{(A+B)\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + (A-B)\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{(A+B)\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + (A-B)\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} + \omega\lambda\sqrt{-\frac{\alpha_2}{2\alpha_3}}\exp[i(lx - vt)] \tag{34}$$

and

$$\psi_{2_2} = \left(-\omega\sqrt{\frac{-2\alpha_2}{\alpha_3}}\,\frac{-\lambda + \sqrt{\lambda^2 - 4\mu}}{2} \right) \times \exp[i(lx - vt)] \tag{35}$$

$$\frac{(A+B)\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + (A-B)\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{(A+B)\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + (A-B)\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} - \omega\lambda\sqrt{-\frac{\alpha_2}{2\alpha_3}}\exp[i(lx - vt)] \tag{36}$$

The dark 1-soliton solution is obtained, when $\lambda^2 - 4\mu \geq 0$ and $A = B = 1$ is considered.

$$\psi_{1_{21}} = -\psi_{2_{21}} = \omega\sqrt{\frac{\alpha_2(4\mu - \lambda^2)}{2\alpha_3}} \times \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)\exp[i(lx - vt)] \tag{37}$$

The singular 1-soliton solution is obtained, when $\lambda^2 - 4\mu \geq 0$ and $A = -B = 1$ was considered.

$$\psi_{1_{22}} = -\psi_{2_{22}} = \omega\sqrt{\frac{\alpha_2(4\mu - \lambda^2)}{2\alpha_3}} \times \coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)\exp[i(lx - vt)] \tag{38}$$

As a byproduct, a singular periodic solution is obtained, when $\lambda^2 - 4\mu < 0$ and $A = B = 1$ was considered.

$$\psi_{1_{23}} = -\psi_{2_{23}} = -\omega\sqrt{\frac{\alpha_2(\lambda^2 - 4\mu)}{2\alpha_3}} \times \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\exp[i(lx - vt)] \tag{39}$$

Another singular periodic solution is obtained, when $\lambda^2 - 4\mu < 0$ and $A = -B = 1$ was considered.

$$\psi_{1_{24}} = -\psi_{2_{24}} = \omega\sqrt{\frac{\alpha_2(\lambda^2 - 4\mu)}{2\alpha_3}} \times \cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\exp[i(lx - vt)] \tag{40}$$

It needs to be noted that these singular periodic solutions have no implications in nonlinear optics. These solutions, however, naturally emerge as a by-product from the analysis, depending on the domain of the argument of the function.

5. Conclusion

This paper obtained dark and singular 1-soliton solutions to the model for dual-core optical fibers. The integration mechanism that was adopted is G'/G-expansion scheme. This method however fails to retrieve bright 1-soliton solutions to the model. This is a limitation of the integration scheme where bright soliton solutions cannot be recovered. As a by-product, singular periodic solutions are obtained although these are not applicable in nonlinear fiber optics.

This paper thus provides a lot of encouragement for future research in this area. Later, additional integration schemes will be applied to obtain bright, dark and singular soliton solutions to the model. Additionally, the model will be considered with other forms of nonlinear media. These are parabolic law, saturable law, polynomial law, log law and several others. The results of that research will be reported in near future.

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