

Optical solitons with quadratic nonlinearity and spatio-temporal dispersion

M. SAVESCU^{a,b}, E. M. HILAL^c, A. A. ALSHAERY^c, A. H. BHRAWY^{d,e}, L. MORARU^f, A. BISWAS^{a,d,*}

^aDepartment of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA

^bDepartment of Mathematics, Kutztown University of Pennsylvania, 15200 Kutztown Road, Kutztown, PA-19530, USA

^cDepartment of Mathematics, Faculty of Science for Girls, King Abdulaziz University, Jeddah, Saudi Arabia

^dDepartment of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah-21589, Saudi Arabia

^eDepartment of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

^fDepartment of Chemistry, Physics and Environment University Dunarea de Jos Galati, 111 Domneasca Street, 800201 Galati, Romania

This paper studies bright, dark and singular soliton solutions to quadratic nonlinear media in presence of spatio-temporal dispersion as well as inter-modal dispersion. Exact 1-soliton solution is obtained. There are several constraints that will naturally fall out during the course of derivation of the soliton solution. These constraint conditions must hold in order for the solitons to exist.

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1. Introduction

Optical solitons is one of the major areas of research in the field of optoelectronics and nonlinear optics [1-25]. There has been an overwhelming number of results that are reported so far in this field. Integrability aspects, perturbation theory, experimental results and other engineering aspects are common grounds of study in this context. The most common types of nonlinear media that are addressed in this context are non-Kerr law and there are various types of such nonlinear forms. It is about time to change gear and focus on another form of nonlinearity. This is the quadratic nonlinear media. The integrability aspect will be the focus of this paper with an aim to extract bright, dark and singular 1-soliton solution.

With quadratic nonlinearity, the second harmonic generation (SHG) represents the nonlinear effect. The pump wave at fundamental harmonic (FH) generates the second harmonic (SH). SHG is derivable from Maxwell's equation with quadratic nonlinearity. Applicability of solitons in quadratic nonlinear media is wide ranged. It is applicable in optical routing and switching along with quadratic nonlinear crystals. In fact, optical solitons with quadratic nonlinearity has also been experimentally observed [24].

2. Governing equation

For quadratic nonlinear media, with inter-modal dispersion (IMD) and spatio-temporal dispersion (STD) is given by

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 q + k_1 q^* r = i\alpha_1 q_x \quad (1)$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 r + k_2 q^2 = i\alpha_2 r_x \quad (2)$$

Here, $q(x, t)$ and $r(x, t)$ represents the wave profile of the FH and SH components respectively. The independent variables are x and t that are spatial and temporal variables. The coefficients of group velocity dispersion (GVD) terms are a_l with $l = 1; 2$ for the two components. Then, the coefficients of STD are b_l . The coefficients of the quadratic nonlinear terms are k_l while the IMD terms are on the right hand sides of the two components and are given by the coefficients of α_l . It was pointed out during 2011 that the inclusion of the STD makes the governing NLSE well-posed as opposed to the consideration of GVD alone, in which case, the model problem stays ill-posed [15, 18]. The first term for both components is the linear evolution.

There has been a lot of studies carried out in the past on quadratic nonlinear media and consequently lots of results are reported during the past couple of decades [1-5, 7-13, 22-25]. Very recently, exact bright and singular 1-soliton solution was obtained for quadratic nonlinear media in presence of GVD only and also without IMD [1]. This paper is thus an extension and generalization of these earlier reports. The ansatz approach will be the integration tool adopted in this paper. Bright, dark and singular solitons will be obtained in this paper along with several constraint conditions that must hold for the solitons to exist.

3. Soliton solutions

In order to proceed with the ansatz approach, the method is first described in its simplest form. First a solution guess is made for bright, dark and singular soliton. This guess is subsequently substituted into the governing equation and the results fall out. The constraint conditions and other integrability conditions are natural consequences during the process. Therefore the starting hypothesis is [1, 19-21]

$$q(x,t) = P_1(x,t)e^{i\phi(x,t)} \quad (3)$$

$$r(x,t) = P_2(x,t)e^{2i\phi(x,t)} \quad (4)$$

where $P_l(x,t)$ ($l = 1, 2$) represents the amplitude component of the soliton and $\phi(x,t)$ gives the phase component with

$$\phi(x,t) = -\kappa x + \omega t + \theta \quad (5)$$

Here, κ is the soliton frequency, ω is the wave number and θ is the phase constant. Substituting (3), (4) and (5) into (1) and (2) and decomposing into real and imaginary parts gives

$$P_1(\omega + a_1\kappa^2 - b_1\kappa\omega + \alpha_1\kappa - c_1) - a_1 \frac{\partial^2 P_1}{\partial x^2} - b_1 \frac{\partial^2 P_1}{\partial x \partial t} - k_1 P_1 P_2 = 0 \quad (6)$$

and

$$v = \frac{2a_1\kappa - b_1\omega + \alpha_1}{b_1\kappa - 1} \quad (7)$$

respectively, from the first component, where v is the speed of the soliton. From the second component, one recovers

$$P_2(2\omega + 4a_2\kappa^2 - 4b_2\kappa\omega + 2\alpha_2\kappa - c_2) - a_2 \frac{\partial^2 P_1}{\partial x^2} - b_2 \frac{\partial^2 P_1}{\partial x \partial t} - k_2 P_1^2 = 0 \quad (8)$$

and

$$v = \frac{4a_2\kappa - 2b_2\omega + \alpha_2}{2b_2\kappa - 1} \quad (9)$$

Equating the speed of the solitons of the two components from (7) and (9) gives

$$4\kappa^2(a_1b_2 - a_2b_1) + \kappa\{2\alpha_1b_2 - \alpha_2b_1 - 2(a_1 - 2a_2)\} + \omega(b_1 - 2b_2) + (\alpha_2 - \alpha_1) = 0 \quad (10)$$

Setting the coefficients of independent parameters ω and κ leads to

$$b_1 = 2b_2 \quad (11)$$

$$\alpha_1 = \alpha_2 \quad (12)$$

$$a_1 = 2a_2 \quad (13)$$

Therefore, it makes sense to define

$$b_1 = 2b \quad \text{and} \quad b_2 = b \quad (14)$$

$$\alpha_1 = \alpha_2 = \alpha \quad (15)$$

$$a_1 = 2a \quad \text{and} \quad a_2 = a \quad (16)$$

Consequently, speed of the solitons for both components reduce to

$$v = \frac{4a\kappa - 2b\omega + \alpha}{2b\kappa - 1} \quad (17)$$

Also, real part components modify to

$$P_1(\omega + a\kappa^2 - 2b\kappa\omega + \alpha\kappa - c_1) - 2a \frac{\partial^2 P_1}{\partial x^2} - 2b \frac{\partial^2 P_1}{\partial x \partial t} - k_1 P_1 P_2 = 0 \quad (18)$$

and

$$P_2(2\omega + 4a\kappa^2 - 4b\kappa\omega + 2\alpha\kappa - c_2) - a \frac{\partial^2 P_1}{\partial x^2} - b \frac{\partial^2 P_1}{\partial x \partial t} - k_2 P_1^2 = 0 \quad (19)$$

respectively. Additionally, the governing model equations (1) and (2) simplify to

$$iq_t + 2aq_{xx} + 2bq_{xt} + c_1q + k_1q^*r = i\alpha q_x \quad (20)$$

$$ir_t + ar_{xx} + br_{xt} + c_2r + k_1q^2 = i\alpha r_x \quad (21)$$

These equations will now be analytically solved for bright, dark and singular soliton in the following three subsections based on the preliminary analysis that is carried out so far.

3.1 Bright solitons

For bright solitons, the following hypothesis is selected [1, 19-21]

$$P_l = A_l \operatorname{sech}^{p_l} \tau \quad (22)$$

for $l = 1, 2$ where

$$\tau = B(x - vt) \quad (23)$$

Here A_l are the amplitudes of the solitons in the two components and B is the inverse width of these solitons. The values of the unknown exponents p_l will be determined. Substituting (22) into (18) and (19) leads to

$$2B^2 p_1 (p_1 + 1)(bv - a)\operatorname{sech}^2 \tau + k_1 A_2 \operatorname{sech}^{p_1} \tau + (c_2 - 2c_1)(3b\alpha + 3a + 2b^2 c_1 - 4b^2 c_2) > 0 \quad (36)$$

$$+ \{\omega(2b\kappa - 1) - 2a\kappa^2 + c_1 - \alpha\kappa - 2p_1^2 B^2 (bv - a)\} = 0 \quad (24)$$

and

$$A_2 B^2 p_2 (p_2 + 1)(bv - a)\operatorname{sech}^{p_2+2} \tau + k_2 A_1^2 \operatorname{sech}^{2p_1} \tau + A_2 \{2\omega(2b\kappa - 1) - 4a\kappa^2 + c_2 - 2\alpha\kappa - 2p_2^2 B^2 (bv - a)\} \operatorname{sech}^{p_2} \tau = 0 \quad (25)$$

respectively. Next, balancing principle yields

$$p_1 = p_2 = 2 \quad (26)$$

From (24), setting the coefficients of linearly independent functions to zero implies

$$v = \frac{12aB^2 - k_1 A_2}{12bB^2} \quad (27)$$

and

$$\omega = \frac{6a\kappa^2 - 3c_1 + 3\alpha\kappa - 2k_1 A_2}{3(2b\kappa - 1)} \quad (28)$$

Similarly, from (25), setting the coefficients of the linearly independent functions give

$$v = \frac{6aA_2 B^2 - k_2 A_1^2}{6bA_2 B^2} \quad (29)$$

and

$$\omega = \frac{12aA_2 \kappa^2 - 3c_2 A_2 + 6\alpha\kappa A_2 - 2k_2 A_1^2}{6A_2 (2b\kappa - 1)} \quad (30)$$

Equating the speed of the solitons from (27) and (29) leads to the ratio of the amplitudes being given by

$$\frac{A_1}{A_2} = \sqrt{\frac{k_1}{2k_2}} \quad (31)$$

with the constraint

$$k_1 k_2 > 0 \quad (32)$$

Next, equating the wave numbers from (28) and (30) leads to

$$A_1 = \frac{c_2 - 2c_1}{\sqrt{2k_1 k_2}} \quad (33)$$

and

$$A_2 = \frac{c_2 - 2c_1}{k_1} \quad (34)$$

after implementing (31). Finally, equating the speed of the solitons from (17) and (27) yields

$$B = \frac{2b\kappa - 1}{2} \sqrt{\frac{c_2 - 2c_1}{3b\alpha + 3a + 2b^2 c_1 - 4b^2 c_2}} \quad (35)$$

that poses the constraint condition

that must remain valid in order for bright solitons to exist. Incidentally, upon equating the speed of the solitons from (17) and (29) also leads to same width of the soliton given by (35). Finally, bright 1-soliton solutions for the two components are

$$q(x, t) = A_1 \operatorname{sech}^2 [B(x - vt)] e^{i(-\kappa x + \omega t + \theta)} \quad (37)$$

$$r(x, t) = A_2 \operatorname{sech}^2 [B(x - vt)] e^{2i(-\kappa x + \omega t + \theta)} \quad (38)$$

3.2 Dark solitons

For dark soliton solution, the starting hypothesis is given by [13]

$$P_l = A_l + B_l \tanh^{p_l} \tau \quad (39)$$

for $l = 1, 2$. Here A_l , B_l and B are free parameters. The definition of where τ is given by (23). Substituting this hypothesis into (18) and (19) reduces them to

$$p_1 (p_1 + 1)(a - bv)B_1 B^2 \tanh^{p_1+2} \tau - \{B_1(\omega + 2a\kappa^2 - 2b\kappa\omega + \alpha\kappa - c_1) + 4(a - bv)p_1^2 B_1 B^2 - k_1 B_1 A_2\} \tanh^{p_1} \tau + 2p_1 (p_1 - 1)(a - bv)B_1 B^2 \tanh^{p_1-2} \tau + k_1 A_1 B_2 \tanh^{p_2} \tau + k_1 B_1 B_2 \tanh^{p_1+p_2} \tau - A_1(\omega + 2a\kappa^2 - 2b\kappa\omega + \alpha\kappa - c_1 - k_1 A_2) = 0 \quad (40)$$

and

$$p_2 (p_2 + 1)(a - bv)B_2 B^2 \tanh^{p_2+2} \tau - \{B_2(2\omega + 4a\kappa^2 - 4b\kappa\omega + 2\alpha\kappa - c_2) + 2(a - bv)p_2^2 B_2 B^2\} \tanh^{p_2} \tau + p_2 (p_2 - 1)(a - bv)B_2 B^2 \tanh^{p_2-2} \tau + k_2 B_1^2 \tanh^{2p_1} \tau - A_2(2\omega + 4a\kappa^2 - 4b\kappa\omega + 2\alpha\kappa - c_2 - k_2 A_1^2) = 0 \quad (41)$$

Balancing principle again yields (26). The coefficients of linearly independent functions yield

$$v = \frac{12aB^2 + k_1 B_2}{12bB^2} \quad (42)$$

and

$$\omega = \frac{3B_1(2a\kappa^2 + \alpha\kappa - c_1) - k_1(4B_1 B_2 + 3B_1 A_2 + 3A_1 B_2)}{3B_1(2b\kappa - 1)} \quad (43)$$

from the first component (40). The coefficients of linearly independent functions from second component (41) gives

$$v = \frac{6aB_2 B^2 + k_2 B_1^2}{6bB_2 B^2} \quad (44)$$

and

$$\omega = \frac{3B_2(4a\kappa^2 + 2\alpha\kappa - c_2) - 2k_2(2B_1 + 3A_1)}{6B_2(2b\kappa - 1)} \quad (45)$$

Next equating the speed of dark solitons from (42) and (44) leads to the ratio of free parameters B_l for $l = 1, 2$ as

$$\frac{B_1}{B_2} = \sqrt{\frac{k_1}{2k_2}} \quad (46)$$

which again prompts the constraint given by (32). Next, setting the wave numbers, from (43) and (45), equal to one another yields the algebraic relation between the free parameters as

$$3B_1B_2(2c_1 - c_2 + 2k_1A_2) - 4k_2B_1^3 - 6A_1k_2B_1^2 + 8k_1B_1B_2^2 + 6k_1A_1B_2^2 = 0 \quad (47)$$

Now, equating the speed of the soliton from (17) and (42) leads to the free parameter B as

$$B = \frac{1}{2} \left[\frac{k_1B_2(2b\kappa - 1)}{3(2ab\kappa - 2b^2\omega + b\alpha + a)} \right]^{\frac{1}{2}} \quad (48)$$

that introduces the constraint condition

$$k_1B_2(2b\kappa - 1)(2ab\kappa - 2b^2\omega + b\alpha + a) > 0 \quad (49)$$

Similarily equating the speed of the soliton from (17) and (44) also yields (48) and hence (49).

Now, substituting the wave number ω from (43) into (48) gives

$$B = \frac{2b\kappa - 1}{2} \times \left[\frac{k_1B_1B_2}{3B_1(2b^2c_1 - a - \alpha b) + 2k_1b^2(4B_1B_2 + 3B_1A_2 + 3A_1B_2)} \right]^{\frac{1}{2}} \quad (50)$$

with the constraint

$$k_1B_1B_2\{3B_1(2b^2c_1 - a - \alpha b) + 2k_1b^2(4B_1B_2 + 3B_1A_2 + 3A_1B_2)\} > 0 \quad (51)$$

From real part equations (40) and (41) subtracting the constant terms leads to

$$2vbB^2(B_2 - B_1) + \omega(2b\kappa - 1)(A_1 - 2A_2) - 2a\kappa^2(A_1 - 2A_2) - \alpha\kappa(A_1 - 2A_2) + \{2aB^2(2B_1 - B_2) + c_1A_1 - c_2A_2 + k_1A_1A_2 - k_2A_1^2\} = 0 \quad (52)$$

This leads to the conclusion, from the coefficients of independent parameters, and after implementing (46) and (47)

$$A_1 = 2A_2 = \frac{2c_1 - c_2}{3k} \quad (53)$$

and

$$B_2 = 2B_1 = \frac{3(c_2 - 2c_1)}{k} \quad (54)$$

where

$$2k_1 = k_2 = 2k \quad (55)$$

Hence, for dark 1-soliton NLSE with quadratic nonlinearity, the model equations (20) and (21) further simplify to

$$iq_t + 2aq_{xx} + 2bq_{xt} + c_1q + kq^*r = i\alpha q_x \quad (56)$$

$$ir_t + ar_{xx} + br_{xt} + c_2r + 2kq^2 = i\alpha r_x \quad (57)$$

whose dark 1-soliton solution is:

$$q(x, t) = (A_1 + B_1 \tanh^2 \tau) e^{i(-\kappa x + \omega t + \theta)} \quad (58)$$

$$r(x, t) = (A_1 + B_1 \tanh^2 \tau) e^{i(-\kappa x + \omega t + \theta)} \quad (59)$$

where the free parameters, speed and wave numbers are explicitly determined.

3.3 Singular solitons

For singular solitons, the starting hypothesis is [1, 19-21]

$$P_l = A_l \operatorname{csch}^{p_l} \tau \quad (60)$$

for $l = 1, 2$. For singular solitons, A_l and B are still free parameters as in dark optical solitons. In this case, substituting (60) into (18) and (19) simplifies them to

$$2B^2(a - bv)p_1(p_1 + 1)\operatorname{csch}^{p_1+2} \tau + k_1A_2\operatorname{csch}^{p_1+p_2} \tau + \{(\omega(2b\kappa - 1) - 2a\kappa^2 - \alpha\kappa + c_1)\}\operatorname{csch}^{p_1} \tau + 2B^2(a - bv)p_1\operatorname{csch}^{p_1} \tau = 0 \quad (61)$$

and

$$2A_2B^2(a - bv)p_2(p_2 + 1)\operatorname{csch}^{p_2+2} \tau + k_2A_1^2\operatorname{csch}^{2p_1} \tau + \{(2\omega(2b\kappa - 1) - 4a\kappa^2 - 2\alpha\kappa + c_2)\}\operatorname{csch}^{p_2} \tau + B^2(a - bv)p_2\operatorname{csch}^{p_2} \tau = 0 \quad (62)$$

respectively. Balancing principle again yields (26). Then, from the coefficients of the linearly independent functions (61) and (62) gives

$$v = \frac{12aB^2 + k_1A_2}{12bB^2} \quad (63)$$

$$\omega = \frac{6a\kappa^2 - 3c_1 + 3\alpha\kappa + 2k_1A_2}{3(2b\kappa - 1)} \quad (64)$$

and

$$v = \frac{6aA_2B^2 + k_2A_1^2}{6bA_2B^2} \quad (65)$$

$$\omega = \frac{12aA_2\kappa^2 - 3c_2A_2 + 6\alpha\kappa A_2 + 2k_2A_1^2}{6A_2(2b\kappa - 1)} \quad (66)$$

Equating the speed of the solitons from (63) and (65) leads to (31) and (32). Next, equating the wave numbers from (64) and (66) gives

$$A_1 = \frac{2c_1 - c_2}{\sqrt{2k_1k_2}} \quad (67)$$

and

$$A_2 = \frac{2c_1 - c_2}{k_1} \quad (68)$$

with the utilization of (31). Finally, equating the speed of the solitons from (17) and (63) yields (35) and (36). Therefore, singular 1-soliton for quadratic nonlinear media with STD is

$$q(x, t) = A_1 \operatorname{csch}^2[B(x-vt)] e^{i(-\kappa x + \omega t + \theta)} \quad (69)$$

$$r(x, t) = A_2 \operatorname{csch}^2[B(x-vt)] e^{2i(-\kappa x + \omega t + \theta)} \quad (70)$$

These singular solitons exist as long as the appropriate constraints are in place.

4. Conclusions

This paper obtained exact 1-soliton solution to the NLSE with quadratic nonlinearity in presence of STD as well as IMD. Bright, dark and singular soliton solutions are obtained along with necessary constraint conditions that must hold for solitons to exist. The results of this paper are going to be extremely useful to conduct further research. One needs to consider time-dependent coefficients. Later, additional tools of integrability will be applied in order to study this equation further along such as Lie symmetry analysis, G'/G -expansion approach, exp-function method and several others. Some additional aspects of research in this avenue are establishing the quasi-particle theory, computation of quasi-stationary soliton solution and several others. These form the tip of the iceberg.

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References

- [1] A. A. Alshaery, A. H. Bhrawy, E. M. Hilal, A. Biswas, *J Electromagnet. Wave.* **28**, 275 (2014).
- [2] F. K. Asadi, B. Shokri, H. Leblond, *Opt. Commun.* **294**, 283 (2013).
- [3] G. Assanto, G. I. Stegeman, *Opt. Express.* **10**, 388 (2002).
- [4] O. Bang, C. B. Clausen, P. Christiansen. *Opt. Lett.* **24**, 1413 (1999).
- [5] L. Berge, V. K. Mezentsev, J. J. Rammussen, J. Wyller, *Phys. Rev. A.* **52**, R28 (1995).
- [6] A. Biswas, K. Khan, A. Rahman, A. Yildirim, T. Hayat, O. M. Aldossary, *J Optoelectron. Adv. Mater.* **14**, 571 (2012).
- [7] A. V. Buryak, Y. S. Kivshar. *Opt. Lett.* **19**, 1612 (1994).
- [8] A. V. Buryak, Y. S. Kivshar, S. Trillo, *Opt. Lett.* **20**, 1961 (1995).
- [9] A. Buryak, P. D. Trapani, D. V. Skryabin, S. Trillo, *Phys. Rep.* **370**, 63 (2002).
- [10] A. D. Capobianco, B. Costantini, C. D. Angelis, A. L. Palma, G. F. Nalesso, *J Opt. Soc. Am. B.* **14**, 2602 (1997).
- [11] M. Conforti, F. Baronio, C. De Angelis, *IEEE Photonics J.* **2**, 600 (2010).
- [12] M. Conforti, *Opt. Lett.* **39**, 2427 (2014).
- [13] X-J. Deng, *Chinese J Phys.* **46**, 511 (2008).
- [14] I. Dolev, A. Libster, A. Arie, *Appl. Phys. Lett.* **101**, 101109 (2012).
- [15] X. Geng, Y. Lv, *Nonlinear Dynam.* **69**, 1621(2012).
- [16] C. Hang, V. V. Konotop, B. A. Malomed, *Phys. Rev. A.* **80**, 023824 (2009).
- [17] K. Hayata, M. Koshiba, *Phys. Rev. Lett.* **71**, 3275 (1993).
- [18] S. Kumar, K. Singh, R. K. Gupta, *Pramana.* **79**, 41 (2012).
- [19] D. Milovic, A. Biswas, *Serbian J Electric. Eng.* **10**, 365 (2013).
- [20] M. Savescu, K. R. Khan, P. Naruka, H. Jafari, L. Moraru, A. Biswas, *J Comput. Theor. Nanos.* **10**, 1182 (2013).
- [21] M. Savescu, K. R. Khan, R. W. Kohl, L. Moraru, A. Yildirim, A. Biswas, *J Nanoelectron Optoe.* **8**, 208 (2013).
- [22] A. Shapira, N. Voloch-Boloch, B. A. Malomed, A. Arie, *J Opt. Soc. Am. B.* **28**, 1481 (2011).
- [23] L. Torner, A. Barthelemy, *IEEE J Quantum Elect.* **39**, 22 (2003).
- [24] W. E. Tourruellas, Z. Wang, D. J. Hagan, E. W. VanStryland, G. I. Stegeman, L. Torner, C. R. Menyuk, *Phys. Rev. Lett.* **74**, 5036 (1995).
- [25] F. W. Wise, *Pramana.* **57**, 1129 (2001).

*Corresponding author: biswas.anjan@gmail.com