Optical solitons with resonant nonlinear Schrodinger's equation using three integration schemes

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This paper integrates resonant nonlinear Schrödinger equation (RNLSE) with power law nonlinearity and time dependent coefficients. The first integral method (FIM) is applied to reach the optical solitons of RNLSE with power law nonlinearity and time dependent coefficients which are the terms of velocity dispersion, linear and nonlinear terms and also resonant one.

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1. Introduction

The dynamics of optical solitons propagating through optical fibers for trans-continental and trans-oceanic distances is governed by the nonlinear Schrödinger's equation (NLSE). This NLSE is derived from Maxwell's equation in electromagnetic by the aid of multiple-scale perturbation analysis. The NLSE appears, in the literature of optical solitons, with several forms of nonlinearity that depends on the context where it is studied. The most known mathematical modeling of optical systems generally is expressed by types of NLSE. The details of NLSE are given in the studies on nonlinear optics [1-50].

It is crucial to reach general solutions of these corresponding nonlinear equations. Thus, the general solutions of these equations provide much information about the character and the structure of solitons that governs the technological advances in telecommunications industry. Many effective methods have been improved to provide much information for scientists and engineers. Some of these methods are extended tanh, *G'/G*-expansion, Jacobi elliptic function, functional variable, *F*-expansion, ansatz approach, first integral, Kudryashov, and trial equation methods [1-50]. All of these methods are effective methods for acquiring traveling wave solutions NPDE.

The FIM initially has been successfully applied earlier, to solve Burgers-KdV equation, by Feng [16]. This method has also been successfully implemented in various forms of nonlinear evolution equations, including fractional evolution equations that resulted in the retrieval of a spectrum of novel solutions. During recent years, many studies on this method have been made. Raslan [39] has used this method for the Fisher equation. Tascan and Bekir [43] have used this method for Cahn-Allen equation. Abbasbandy and Shirzadi [1] have investigated Benjamin Bona-Mohany equation by this method. Jafari et al. [23] and K. Hosseini et al. [24] has researched w.r.t for Biswas-Milovic equation and KP equation so on [22], [41].

This paper discusses a version of NLSE that is known as resonant NLSE (RNLSE) which also yields soliton solutions. There are three integration schemes that are applied here. They are first integral method, generalized Kudryashov's scheme and the extended trial equation algorithm. These are schematically described in the following three sections and are successfully applied to RNLSE with various nonlinear forms. Soliton and other solutions therefore successfully emerged.

2. Governing Equation

RNLSE [14], [15], [33-37] in dimensionless form [45] with time dependent coefficients is

$$ih_t + a(t)h_{xx} + b(t)F\left(|h|^2\right)h + c(t)\left(\frac{|h|_{xx}}{|h|}\right)h = id(t), \tag{1}$$

where the terms $a(t), b(t), c(t)$ and $d(t)$ are group velocity dipersion, nonlinear, resonant and also linear attenuation, respectively. We will discuss the equation (1) in the power law nonlinearity, which seems in the circumstances that form the setting for plasma physics, turbulance theory and nonlinear fiber optics, as following:

$$ih_t + a(t)h_{xx} + b(t)|h|^{2m} h + c(t)\left(\frac{|h|_{xx}}{|h|}\right)h = 0 \tag{2}$$

such that $\xi = x \mp ct$ and $Q' = \partial Q(\xi)/\partial\xi.$
where m is the nonlinear parameter.

3. First integral method

The proposed method can be summarized in the following steps:

Step 1. The common nonlinear partial differential equation NPDE:

$$W(h, h_t, h_x, h_{xt}, h_{tt}, h_{xx}, \ldots) = 0, \tag{3}$$

using a wave variable $\xi = x \mp ct$ transforms to the ordinary differential equation (ODE) as

$$L(H, H', H'', H''', \ldots) = 0 \tag{4}$$

such that $H' = \partial H(\xi)/\partial\xi.$

Step 2. The solution of ODE (4) can be written as:

$$h(x,t) = h(\xi) \tag{5}$$

Step 3. Taking the following independent variables as

$$H(\xi) = h(\xi), G(\xi) = \partial h(\xi)/\partial\xi \tag{6}$$

a new system of ODEs are given by

$$\begin{aligned} &\partial H(\xi)/\partial\xi = G(\xi) \\ &\partial F(\xi)/\partial\xi = P(H(\xi), G(\xi)) \end{aligned} \tag{7}$$

Step 4. Due to lack of systematic theory which gives us some methods for finding first integrals, we will apply the Division Theorem (DT) to obtain the integrals (7). This will reduce (2) to a first-order integrable ODE. Finally, an exact solution to (1) is obtained by solving this equation.

3.1. Application to R-NLSE

We use the following transformation

$$h = H(\xi)e^{i\left[-\alpha x + \int v(t)dt\right]}, \quad \xi = \beta x + \int w(t)dt$$

and get ODE system

$$(w - 2a\alpha\beta)H_\xi = 0, \tag{8}$$

$$\left(v + a\alpha^2\right)H - bH^{2m+1} - (a+c)\beta^2 H_{\xi\xi} = 0 \tag{9}$$

Then, solving (8) we get

$$w = 2a\alpha\beta \tag{10}$$

Further, we balance the term H^{2m+1} with $H_{\xi\xi}$, and substitute the transformation $H = Q^{\frac{1}{m}}$ into (9) and obtain:

$$\begin{aligned} &\left(v + a\alpha^2\right)m^2Q^2 - m^2bQ^4 + \\ &(m-1)(a+c)\beta^2Q_\xi^2 - m(a+c)\beta^2 QQ_{\xi\xi} = 0, \end{aligned} \tag{11}$$

By using transformation $Q_\xi = G,$ we have

$$G_\xi = \frac{\left(v + a\alpha^2\right)m^2Q^2 - m^2bQ^4}{m(a+c)\beta^2 Q} + \frac{(m-1)(a+c)\beta^2G^2}{m(a+c)\beta^2 Q}. \tag{12}$$

Assuming that $d\xi = Qd\tau$ we get

$$\begin{aligned} &Q_\tau = QG, \\ &G_\tau = \frac{\left(v + a\alpha^2\right)mQ^2}{(a+c)\beta^2} - \frac{mbQ^4}{(a+c)\beta^2} + \frac{G^2}{m} \end{aligned} \tag{13}$$

Further, it is supposed that $H(\tau)$ and $G(\tau)$ are non-trivial solutions of Eq. (13) and $F(Q,G) = \sum_{i=0}^{r} a_i(Q)G^i$ is an irreducible function in the domain $C[Q,G]$ satisfying

$$F(Q(\tau), G(\tau)) = \sum_{i=0}^{r} a_i(Q)G^i = 0 \tag{14}$$

where $a_i(Q)$, $(i = 0,1,2,\ldots,r)$ are polynomials of Q and $a_r(Q) \neq 0$. Eq.(12) is the first integral for system (13), owing to the DT, there exists $g(Q) + h(Q)G$ in $C[Q,G]$ as:

$$dF/d\tau = \frac{dF}{dQ}\frac{dQ}{d\tau} + \frac{dF}{dG}\frac{dG}{d\tau} = [g(Q)+h(Q)G]\sum_{i=0}^{r} a_i(Q)G^i \tag{15}$$

Considering $r=1$ in Eq. (15) and equating the coefficients of $G^i (i=0,1,2,\dots,r)$ of Eq. (15, we have

$$\dot{a}_1(Q)Q = a_1(Q)\left(h(Q) - \frac{1}{m}\right) \tag{16}$$

$$\dot{a}_0(Q)Q = a_1(Q)g(Q) + h(Q)a_0(Q) \tag{17}$$

$$a_0(Q)g(Q) = a_1(Q)\left[\frac{\left(v+a\alpha^2\right)mQ^2}{(a+c)\beta^2} - \frac{mbQ^4}{(a+c)\beta^2}\right] \tag{18}$$

Since $a_i(Q)(i=0,1)$ is polynomial of $Q, a_1(Q)$ is a constant and $h(Q)=\frac{1}{m}$ from (14). For convenience, it is obtained $a_1(Q)=1$. By equalization the degrees of $g(Q)$ and $a_0(Q)$ we conclude the degree of $g(Q)$ is equal to two. Then, we assume that $g(Q)=G_0+G_1Q+G_2Q^2$, we obtain from Eq. (17) as follows

$$a_0(Q) = A_2Q^2 + A_1Q + A_0 \tag{19}$$

Replacing $a_0(Q)$, $a_1(Q)$ and $g(Q)$ in Eq. (18) to separate the common factor of the same terms, then equating the coefficients of Q^i to zero, we have following cases:

$$v = -\frac{aG_2\alpha^2 + bG_0}{G_2}, \quad A_0 = A_1 = G_1 = 0, A_2 = -\frac{mb}{G_2(a+c)\beta^2}. \tag{20}$$

$$v = -\frac{aG_2\alpha^2 + bG_1}{G_2}, A_0 = G_0 = 0, \quad A_1 = \frac{G_1 mb}{G_2^2(a+c)\beta^2}, A_2 = -\frac{mb}{G_2(a+c)\beta^2} \tag{21}$$

$$A_1 = G_0 = G_1 = 0, A_2 = -\frac{mb}{G_2(a+c)\beta^2} \quad A_0 = \frac{am\alpha^2 + mv}{G_2(a+c)\beta^2}. \tag{22}$$

$$A_1 = G_0 = G_1 = 0, A_0 = \frac{a\alpha^2 + v}{G_2}, \quad A_2 = -\frac{mb}{G_2(a+c)\beta^2} \tag{23}$$

setting (20-23) into (14) with respectively, we have

$$Q_\xi = \frac{mb}{G_2(a+c)\beta^2}Q^2(\xi) \tag{24}$$

$$Q_\xi = \frac{-G_1 mb}{G_2^2(a+c)\beta^2}Q(\xi) + \frac{mb}{G_2(a+c)\beta^2}Q^2(\xi) \tag{25}$$

$$Q_\xi = -\frac{am\alpha^2 + mv}{G_2(a+c)\beta^2} + \frac{mb}{G_2(a+c)\beta^2}Q^2(\xi) \tag{26}$$

$$Q_\xi = -\frac{v + a\alpha^2}{G_2} + \frac{mb}{G_2(a+c)\beta^2}Q^2(\xi) \tag{27}$$

If we solve the Eqs. (24-27) with respectively: Firstly we have the following rational solution from Eq. (24)

$$Q(\xi) = -\frac{1}{\frac{mb}{G_2(a+c)\beta^2}\xi + C_0} \tag{28}$$

Where C_0 is constant and the solution of the Eq. (2) with the transformation $H = Q^{\frac{1}{m}}$ and $h = H(\xi)\, e^{i\left[-\alpha x + \int (v) t\right] d}$, $\xi = \beta x + \int w(t)\,dt$.

$$h(x,t) = \left(-\frac{1}{\frac{mb}{G_2(a+c)\beta^2}\left(\beta x + \int w(t)\,dt\right) + C_0}\right)^{\frac{1}{m}} e^{i\left[-\alpha x + \int v(t)\,dt\right]} \tag{29}$$

where $w(t) = 2a(t)\alpha\beta$ and $v = -\frac{aG_2\alpha^2 + bG_0}{G_2}$.
Secondly, we have the following solution from (25)

$$Q(\xi) = \frac{G_1}{G_2 + e^{\frac{G_1 mb}{G_2^2(a+c)\beta^2}\xi + G_1C_0}} = \frac{G_1}{G_2 + \cosh\left[\frac{G_1 mb}{G_2^2(a+c)\beta^2}\xi + G_1C_0\right] + \sinh\left[\frac{G_1 mb}{G_2^2(a+c)\beta^2}\xi + G_1C_0\right]} \tag{30}$$

and the original solution of the Eq. (2) is

$$h(x,t)=\left(\frac{G_1}{G_2+\cosh\left[\frac{G_1 mb}{G_2^2(a+c)\beta^2}\left(\beta x+\int w(t)\,dt\right)+G_1C_0\right]+\sinh\left[\frac{G_1 mb}{G_2^2(a+c)\beta^2}\left(\beta x+\int w(t)\,dt\right)+G_1C_0\right]}\right)^{\frac{1}{m}} e^{i\left[-\alpha x+\int v(t)\,dt\right]}. \tag{31}$$

where $w(t)=2a(t)\alpha\beta$ and $v=-\frac{aG_2\alpha^2+bG_1}{G_2}$.

Thirdly, we have the similar forms for Eqs. (26,27) so we will acquired the one of these equations. For the Eq. (27), we have the following dark solion solution

$$Q=\beta\sqrt{\frac{(a+c)\left(a\alpha^2+v\right)}{bm}}\tanh\left[\beta\sqrt{bm(a+c)\left(a\alpha^2-v\right)}\left(\frac{\xi}{(a+c)G_2\beta^2}+C_0\right)\right] \tag{32}$$

and the original solution of the Eq. (2) is

$$h(x,t)=\left(\beta\sqrt{\frac{(a+c)\left(a\alpha^2-v\right)}{bm}}\tanh\left[\beta\sqrt{b(a+c)\left(a\alpha^2+v\right)}\left(\frac{\beta x+\int w(t)\,dt}{(m-1)(a+c)G_2\beta^2}+C_0\right)\right]\right)^{\frac{1}{m}} e^{i\left[-\alpha x+\int v(t)\,dt\right]}. \tag{33}$$

For another type solution of Eq. (27), if we choose $\frac{v-a\alpha^2}{G_2}=\frac{1}{2}$ and $\frac{mb}{G_2(m-1)(a+c)\beta^2}=-\frac{1}{2}$, then the Eq. (27) becomes

$$Q_\xi=\frac{1}{2}-\frac{1}{2}Q^2(\xi). \tag{34}$$

It is illustrated the solution of the Eq. (34) by Chen and Zhang [41], Eq. (34) has the following solution

$$Q(\xi)=\tanh[\xi]\pm i\,\mathrm{sech}[\xi]. \tag{35}$$

So we have the following dark-brigth optical combo soliton solution of the Eq. (2)

$$q(x,t)=\left(\begin{array}{l}\tanh[\beta x+\int w(t)\,dt]\pm\\ i\,\mathrm{sech}[\beta x+\int w(t)\,dt]\end{array}\right)e^{i\left[-\alpha x+\int v(t)\,dt\right]} \tag{36}$$

where $w(t)=2a(t)\alpha\beta$, $\frac{v-a\alpha^2}{G_2}=\frac{1}{2}$ and $\frac{mb}{G_2(m-1)(a+c)\beta^2}=-\frac{1}{2}$.

4. Generalized Kudryashov's Method

In this section, we describe the generalized Kudryashov method [12] for finding traveling wave solutions of nonlinear partial differential equations (NLPDE) and subsequently will apply this method to solve the R-NLSE.

We suppose that the given NLPDE for $u(x,t)$ is in the form

$$P(u,u_t,u_x,u_{xx},u_{xt},u_{tt},\ldots)=0, \tag{37}$$

where P is a polynomial. The essence of the generalized Kudryashov method can be presented in the following steps:

Step-1: To find the traveling wave solutions of Eq. (37), we introduce the wave variable

$$u(x,t)=U(\xi),\quad \xi=x-vt, \tag{38}$$

where v is a constant to be determined later. Substituting Eq. (38) into Eq. (37), we obtain the following ODE

$$Q(U,U',U'',\ldots)=0. \tag{39}$$

Step-2: Suppose that solution of the Eq. (39) can be written as follows:

$$U(\xi)=\frac{\sum_{i=0}^{N}k_iQ^i(\xi)}{\sum_{j=0}^{M}l_jQ^j(\xi)}=\frac{A[Q(\xi)]}{B[Q(\xi)]}, \tag{40}$$

where k_i $(i=0,1,\ldots,N)$ and l_j $(j=0,1,\ldots,M)$ are constants to be determined later, and $Q(\xi)$ is $1/(1\pm e^{\xi})$. We recall that the function $Q(\xi)$ is solution of equation [26]

$$Q_\xi=Q^2-Q. \tag{41}$$

Taking into consideration (40) along with (41), we have

$$U'(\xi)=(Q^2-Q)\left[\frac{A'B-AB'}{B^2}\right], \tag{42}$$

$$U''(\xi)=\frac{Q^2-Q}{B^2}\left[\begin{matrix}(2Q-1)(A'B-AB')+\\ \frac{Q^2-Q}{B}\left[\begin{matrix}B\left(A''B-AB''\right)\\ -2A'BB'+2A(B')^2\end{matrix}\right]\end{matrix}\right] \quad (43)$$

and so on. Here, the prime denotes the derivative with respect to ξ.

Step-3: Considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (39), We can determine a relation of M and N. We can take some values of M and N.

Step-4: Substituting expressions given by Eqs. (40)-(43) into Eq. (39), we obtain a polynomial $\Lambda(Q)$ of Q. Equating the coefficients of this polynomial to zero, we get a system of algebraic equations. Solving this system, we can find the values of unknown parameters. As a result, we obtain the exact solutions to Eq. (37).

4.1. Application to R-NLSE (Kerr law)

In this section, we apply the generalized Kudryashov method to solve the resonant nonlinear Schrödinger's equation (45)

$$i\psi_t+\alpha\psi_{xx}+\beta F(|\psi|^2)\psi+\gamma\left\{\frac{|\psi|_{xx}}{|\psi|}\right\}\psi=0. \quad (44)$$

The Kerr law nonlinearity is the case when $F(s)=s.$ This kind of nonlinearity typically arises in the context of water waves or nonlinear fiber optics when the refractive index of the light is proportional to the intensity. For Kerr-law nonlinearity, the considered generalized RNLS equation is given by

$$i\psi+\alpha\psi_{xx}+\beta|\psi|^2\psi+\gamma\left\{\frac{|\psi|_{xx}}{|\psi|}\right\}\psi=0. \quad (45)$$

Under the travelling wave transformation

$$\psi(x,t)=U(\xi)e^{i(-\kappa x+\omega t+\theta)},\quad \xi=x+2\kappa\alpha t, \quad (46)$$

we have

$$\left(\alpha+\gamma\right)U''-\left(\omega+\kappa^2\alpha\right)U+\beta U^3=0 \quad (47)$$

We will now analyze Eq. (47) to secure soliton solutions by generalized Kudryashov method. We substitute Eqs. (40) and (43) into Eq. (47). Then, we employ the balance principle and determine a relation of M and N as

$$N=M+1 \quad (48)$$

Case-1: When $M=0$ and $N=1$ in Eq. (48), we have the solution of Eq. (47) in the form

$$U(\xi)=\frac{k_0+k_1Q(\xi)}{l_0} \quad (49)$$

where $k_1\neq 0$ and $l_0\neq 0$. Substituting Eq. (49) into Eq. (47), we have a system of algebraic equations. Solving this system, we find the following results:

$$k_0=\pm\frac{il_0\sqrt{\alpha+\gamma}}{\sqrt{2\beta}},\quad k_1=\mp\frac{il_0\sqrt{2(\alpha+\gamma)}}{\sqrt{\beta}},\quad l_0=l_0,\quad \omega=-\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right] \quad (50)$$

Substituting Eq. (50) along with $Q(\xi)=1/(1\pm e^{\xi})$ into (49), and inserting the result into the wave transformation (46), we obtain the following solitary wave solutions to Eq. (45), respectively:

$$\psi(x,t)=\pm\sqrt{-\frac{\alpha+\gamma}{2\beta}}\tanh\left(\frac{x+2\alpha\kappa t}{2}\right)e^{i\left\{-\kappa x-\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right]t+\theta\right\}}, \quad (51)$$

and

$$\psi(x,t)=\pm\sqrt{-\frac{\alpha+\gamma}{2\beta}}\coth\left(\frac{x+2\alpha\kappa t}{2}\right)e^{i\left\{-\kappa x-\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right]t+\theta\right\}}. \quad (52)$$

Case-2: When $M=1$ and $N=2$ in Eq. (48), we have the solution of Eq. (47) in the form

$$U(\xi)=\frac{k_0+k_1Q(\xi)+k_2Q^2(\xi)}{l_0+l_1Q(\xi)} \quad (53)$$

where $k_2\neq 0$ and $l_1\neq 0$. Substituting Eq. (53) into Eq. (47), we have a system of algebraic equations. Solving this system, we find the following results:

Set-1.

$$k_0=0,\quad k_1=\pm\frac{il_1\sqrt{\alpha+\gamma}}{\sqrt{2\beta}},\quad k_2=\mp\frac{il_1\sqrt{2(\alpha+\gamma)}}{\sqrt{\beta}},\quad l_0=0,\quad l_1=l_1,\quad \omega=-\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right] \quad (54)$$

Set-2.

$$k_0=\pm\frac{il_0\sqrt{\alpha+\gamma}}{\sqrt{2\beta}},\quad k_1=0,\quad k_2=\mp\frac{2il_0\sqrt{2(\alpha+\gamma)}}{\sqrt{\beta}},\quad l_0=l_0,\quad l_1=2l_0,\quad \omega=-\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right] \quad (55)$$

Set-3.

$$k_0 = \pm\frac{il_0\sqrt{\alpha+\gamma}}{\sqrt{2\beta}}, \quad k_1 = \mp\frac{i(2l_0-l_1)\sqrt{\alpha+\gamma}}{\sqrt{2\beta}},$$
$$k_2 = \mp\frac{il_1\sqrt{2(\alpha+\gamma)}}{\sqrt{\beta}}, \quad l_0 = l_0, \quad l_1 = l_1, \tag{56}$$
$$\omega = -\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right]$$

Set-4.

$$k_0 = 0, \quad k_1 = \pm\frac{2il_0\sqrt{2(\alpha+\gamma)}}{\sqrt{\beta}}, \quad k_2 = \mp\frac{2il_0\sqrt{2(\alpha+\gamma)}}{\sqrt{\beta}},$$
$$l_0 = l_0, \quad l_1 = -2l_0, \quad \omega = \alpha+\gamma-\alpha\kappa^2, \tag{57}$$

where κ is arbitrary constant. Consequently, we obtain the following exact traveling wave solutions to the R-NLSE with Kerr law nonlinearity:

By using the results in Eqs. (54)-(56), we find exact 1-soliton solutions as

$$\psi(x,t) = \pm\sqrt{-\frac{\alpha+\gamma}{2\beta}}\tanh\left(\frac{x+2\alpha\kappa t}{2}\right)e^{i\left\{-\kappa x-\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right]t+\theta\right\}}, \tag{58}$$

and

$$\psi(x,t) = \pm\sqrt{-\frac{\alpha+\gamma}{2\beta}}\coth\left(\frac{x+2\alpha\kappa t}{2}\right)e^{i\left\{-\kappa x-\frac{1}{2}\left[\gamma+\alpha\left(1+2\kappa^2\right)\right]t+\theta\right\}}. \tag{59}$$

By using the results in Eq. (57), we have solitary wave solutions as

$$\psi(x,t) = \pm\sqrt{-\frac{2(\alpha+\gamma)}{\beta}}csch\left(x+2\alpha\kappa t\right)e^{i\left\{-\kappa x+\left(\alpha+\gamma-\alpha\kappa^2\right)t+\theta\right\}}, \tag{60}$$

and

$$\psi(x,t) = \pm\sqrt{\frac{2(\alpha+\gamma)}{\beta}}sech\left(x+2\alpha\kappa t\right)e^{i\left\{-\kappa x+\left(\alpha+\gamma-\alpha\kappa^2\right)t+\theta\right\}}. \tag{61}$$

4.2. Application to R-NLSE (Power law)

The power law nonlinearity arises when $F(s)=s^m$, where the parameter m is referred to as the nonlinearity parameter. This kind of law appears in the context of plasma physics, turbulence theory and also sometimes in the case of nonlinear fiber optics. It needs to be however noted that one must have $0<m<2$ in order to avoid self-focusing singularity and soliton collapse [45]. For power law nonlinearity, the R-NLSE takes the form

$$i\psi + \alpha\psi_{xx} + \beta|\psi|^{2n}\psi + \gamma\left\{\frac{|\psi|_{xx}}{|\psi|}\right\}\psi = 0 \tag{62}$$

For searching the one-soliton solution for the above model, we use the same wave transformation

$$\psi(x,t) = U(\xi)e^{i(-\kappa x+\omega t+\theta)}, \quad \xi = x+2\kappa\alpha t. \tag{63}$$

Substituting Eq. (63) into Eq. (62), we obtain ordinary differential equation:

$$\left(\alpha+\gamma\right)U'' - \left(\omega+\kappa^2\alpha\right)U + \beta U^{2n+1} = 0 \tag{64}$$

To obtain an analytic solution, we use the transformation $U = V^{\frac{1}{2n}}$ in Eq. (64) to find

$$\begin{aligned}&(\alpha+\gamma)\left((1-2n)(V')^2+2nVV''\right)\\&-4\left(\omega+\kappa^2\alpha\right)n^2V^2+4\beta n^2V^3 = 0\end{aligned} \tag{65}$$

We will now analyze Eq. (65) to obtain soliton solutions by generalized Kudryashov method. We substitute Eqs. (40), (42) and (43) into Eq. (65). Then, we employ the balance principle and determine a relation of M and N as

$$N = M+2 \tag{66}$$

Case-1: When $M=0$ and $N=2$ in Eq. (66), then we can write the solution of Eq. (65) in the form

$$V(\xi) = \frac{k_0+k_1Q(\xi)+k_2Q^2(\xi)}{l_0} \tag{67}$$

where $k_2 \neq 0$ and $l_0 \neq 0$. Substituting Eq. (67) into Eq. (65), we have a system of algebraic equations. Solving this system, we find the following results:

$$\begin{aligned}&k_0 = 0, \quad k_1 = k_1, \quad k_2 = -k_1, \quad l_0 = \frac{k_1n^2\beta}{(1+n)(\alpha+\gamma)},\\&\omega = -\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2}\end{aligned} \tag{68}$$

Substituting Eq. (68) along with $Q(\xi)=1/(1\pm e^{\xi})$ into Eq. (67) and using the transformation $U = V^{\frac{1}{2n}}$, we obtain exact solution to Eq. (64). Consequently, we have the exact 1-soliton solutions to Eq. (62) as follows:

$$\psi(x,t)=\left\{\frac{(1+n)(\alpha+\gamma)}{4n^2\beta}sech^2\left(\frac{x+2\alpha\kappa t}{2}\right)\right\}^{\frac{1}{2n}} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2}\right)t+\theta\right\}} \quad (69)$$

and

$$\psi(x,t)=\left\{-\frac{(1+n)(\alpha+\gamma)}{4n^2\beta}csch^2\left(\frac{x+2\alpha\kappa t}{2}\right)\right\}^{\frac{1}{2n}} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2}\right)t+\theta\right\}} \quad (70)$$

Case-2: When $M=1$ and $N=3$ in Eq. (66), we have the solution of Eq. (65) in the form

$$V(\xi)=\frac{k_0+k_1Q(\xi)+k_2Q^2(\xi)+k_3Q^3(\xi)}{l_0+l_1Q(\xi)}, \quad (71)$$

where $k_3\neq 0$ and $l_1\neq 0$. Substituting Eq. (71) into Eq. (65), we have a system of algebraic equations. Solving this system, we find the following results:

Set-1.

$$\begin{aligned} &k_0=0,\quad k_1=0,\quad k_2=\frac{l_1(1+n)(\alpha+\gamma)}{n^2\beta},\\ &k_3=-\frac{l_1(1+n)(\alpha+\gamma)}{n^2\beta}, l_0=0,\quad l_1=l_1,\\ &\omega=-\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2} \end{aligned} \quad (72)$$

Set-2.

$$\begin{aligned} &k_0=0,\quad k_1=\frac{l_1(1+n)(\alpha+\gamma)}{n^2\beta},\quad k_2=0,\\ &k_3=-\frac{l_1(1+n)(\alpha+\gamma)}{n^2\beta},\\ &l_0=l_1,\quad l_1=l_1,\quad \omega=-\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2}, \end{aligned} \quad (73)$$

Set-3.

$$\begin{aligned} &k_0=0,\quad k_1=\frac{l_0(1+n)(\alpha+\gamma)}{n^2\beta},\\ &k_2=-\frac{(l_0-l_1)(1+n)(\alpha+\gamma)}{n^2\beta},\\ &k_3=-\frac{l_1(1+n)(\alpha+\gamma)}{n^2\beta},\\ &l_0=l_0,\quad l_1=l_1,\quad \omega=-\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2} \end{aligned} \quad (74)$$

where κ is arbitrary constant. Consequently, we have the exact 1-soliton solutions to the R-NLSE with power law nonlinearity as follows:

$$\psi(x,t)=\left\{\frac{(1+n)(\alpha+\gamma)}{4n^2\beta}sech^2\left(\frac{x+2\alpha\kappa t}{2}\right)\right\}^{\frac{1}{2n}} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2}\right)t+\theta\right\}} \quad (75)$$

and

$$\psi(x,t)=\left\{-\frac{(1+n)(\alpha+\gamma)}{4n^2\beta}csch^2\left(\frac{x+2\alpha\kappa t}{2}\right)\right\}^{\frac{1}{2n}} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\alpha+\gamma}{4n^2}\right)t+\theta\right\}} \quad (76)$$

4.3. Application to R-NLSE (Parabolic law)

For parabolic-law nonlinearity, $F(s)=\beta s+\gamma s^2$, where b and c are in general constants. Such a kind of nonlinearity appears also in fiber optics. In this case, the R-NLSE is

$$i\psi_t+\alpha\psi_{xx}+\left\{\beta|\psi|^2+\gamma|\psi|^4\right\}\psi+d\left\{\frac{|\psi|_{xx}}{|\psi|}\right\}\psi=0 \quad (77)$$

We use the same wave transformation

$$\psi(x,t)=U(\xi)e^{i(-\kappa x+\omega t+\theta)},\quad \xi=x+2\kappa\alpha t \quad (78)$$

Substituting (78) into (77), we obtain ordinary differential equation:

$$(\alpha+d)U''-\left(\omega+\kappa^2\alpha\right)U+\beta U^3+\gamma U^5=0 \quad (79)$$

By using transformation $U=V^{\frac{1}{2}}$, Eq. (79) becomes

$$\begin{aligned} &(\alpha+d)(2VV''-(V')^2)-4\left(\omega+\kappa^2\alpha\right)V^2\\ &+4\beta V^3+4\gamma V^4=0. \end{aligned} \quad (80)$$

We will now analyze Eq. (80) to construct soliton solutions by generalized Kudryashov method. We substitute Eqs. (40), (42) and (43) into Eq. (80). Then, we employ the balance principle and determine a relation of M and N as

$$N=M+1 \quad (81)$$

Case-1: When $M=0$ and $N=1$ in Eq. (81), then we have the solution of Eq. (80) in the form

$$V(\xi)=\frac{k_0+k_1Q(\xi)}{l_0} \quad (82)$$

where $k_1 \neq 0$ and $l_0 \neq 0$. Substituting Eq. (82) into Eq. (80), we obtain a system of algebraic equations. Solving this system, we find the following results:

Set-1.

$$k_0 = 0, \quad k_1 = \frac{l_0(d+\alpha)}{\beta}, \quad l_0 = l_0, \quad \gamma = -\frac{3\beta^2}{4(d+\alpha)}, \quad \omega = \frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right) \tag{83}$$

Set-2.

$$k_0 = \frac{l_0(d+\alpha)}{\beta}, \quad k_1 = -\frac{l_0(d+\alpha)}{\beta}, \quad l_0 = l_0, \quad \gamma = -\frac{3\beta^2}{4(d+\alpha)}, \quad \omega = \frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right) \tag{84}$$

where κ is an arbitrary constant. Substituting Eqs. (83)-(84) along with $Q(\xi) = 1/(1 \pm e^{\xi})$ into Eq. (82) and using the transformation $U = V^{\frac{1}{2}}$, we obtain exact solution to Eq. (79). Consequently, we have the exact 1-soliton solutions to Eq. (77) as follows:
By using the results in Eq. (83), we find

$$\psi(x,t) = \left\{\frac{d+\alpha}{2\beta}\left(1-\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{85}$$

and

$$\psi(x,t) = \left\{\frac{d+\alpha}{2\beta}\left(1-\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{86}$$

By using the results in Eq. (84), we have

$$\psi(x,t) = \left\{\frac{d+\alpha}{2\beta}\left(1+\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{87}$$

and

$$\psi(x,t) = \left\{\frac{d+\alpha}{2\beta}\left(1+\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{88}$$

Case-2: When $M = 1$ and $N = 2$ in Eq. (81), then we have the solution of Eq. (80) in the form

$$V(\xi) = \frac{k_0 + k_1 Q(\xi) + k_2 Q^2(\xi)}{l_0 + l_1 Q(\xi)} \tag{89}$$

where $k_2 \neq 0$ and $l_1 \neq 0$. Substituting Eq. (89) into Eq. (80), we obtain a system of algebraic equations. Solving this system, we find the following results:

Set-1.

$$k_0 = 0, \quad k_1 = \frac{l_0(d+\alpha)}{\beta}, \quad k_2 = \frac{l_1(d+\alpha)}{\beta}, \quad l_0 = l_0, \quad l_1 = l_1, \quad \gamma = -\frac{3\beta^2}{4(d+\alpha)}, \quad \omega = \frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right) \tag{90}$$

Set-2.

$$k_0 = 0, \quad k_1 = \frac{(2l_0+l_1)(d+\alpha)}{\beta}, \quad k_2 = -\frac{(2l_0+l_1)(d+\alpha)}{\beta}, \quad l_0 = l_0, \quad l_1 = l_1, \quad \gamma = -\frac{3l_1^2\beta^2}{4(2l_0+l_1)^2(d+\alpha)}, \quad \omega = \frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right) \tag{91}$$

Set-3.

$$k_0 = \frac{l_0(d+\alpha)}{\beta}, \quad k_1 = -\frac{(l_0-l_1)(d+\alpha)}{\beta}, \quad k_2 = -\frac{l_1(d+\alpha)}{\beta}, \quad l_0 = l_0, \quad l_1 = l_1, \quad \gamma = -\frac{3\beta^2}{4(d+\alpha)}, \quad \omega = \frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right) \tag{92}$$

where κ is an arbitrary constant. Consequently, we obtain the following exact traveling wave solutions to the RNLSE with parabolic-law nonlinearity:

By using the results in Eq. (90), we find exact 1-soliton solutions as

$$\psi(x,t) = \left\{\frac{d+\alpha}{2\beta}\left(1-\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{93}$$

and

$$\psi(x,t) = \left\{\frac{d+\alpha}{2\beta}\left(1-\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{94}$$

By using the results in Eq. (91), we have solitary wave solutions as

$$\psi(x,t)=\left\{\frac{(2l_0+l_1)(d+\alpha)sech^2\left[\frac{x+2\alpha\kappa t}{2}\right]}{2\beta\left(2l_0+l_1-l_1\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)}\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{95}$$

and

$$\psi(x,t)=\left\{-\frac{(2l_0+l_1)(d+\alpha)csch^2\left[\frac{x+2\alpha\kappa t}{2}\right]}{2\beta\left(2l_0+l_1-l_1\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)}\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{96}$$

By using the results in Eq. (92), we obtain exact 1-soliton solutions as

$$\psi(x,t)=\left\{\frac{d+\alpha}{2\beta}\left(1+\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{97}$$

and

$$\psi(x,t)=\left\{\frac{d+\alpha}{2\beta}\left(1+\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2}} e^{i\left\{-\kappa x+\frac{1}{4}\left(d+\alpha-4\alpha\kappa^2\right)t+\theta\right\}} \tag{98}$$

4.4. Application to R-NLSE (Dual-power law)

The dual-power law nonlinearity is formulated as $F(s)=\beta s^n+\gamma s^{2n}$, where b and c are in general constants. This law is a generalization of the parabolic law nonlinearity. In fact, setting $n=1$, the dual-power law collapses to parabolic law nonlinearity. In this case, the R-NLSE is

$$i\psi_t+\alpha\psi_{xx}+\left\{\beta|\psi|^{2n}+\gamma|\psi|^{4n}\right\}\psi+d\left\{\frac{|\psi|_{xx}}{|\psi|}\right\}\psi=0 \tag{99}$$

Without loss of generality, we assume that the solution $\psi(x,t)$ to Eq. (99) takes the form

$$\psi(x,t)=U(\xi)e^{i(-\kappa x+\omega t+\theta)},\quad \xi=x+2\kappa\alpha t \tag{100}$$

Using this the wave transformation, we have

$$(\alpha+d)U''-\left(\omega+\kappa^2\alpha\right)U+\beta U^{2n+1}+\gamma U^{4n+1}=0 \tag{101}$$

To obtain an analytic solution, we propose a transformation denoted by $U=V^{\frac{1}{2n}}$. Then Eq. (101) is converted to

$$\begin{aligned}&(\alpha+d)(2nVV''+(1-2n)(V')^2)\\&-4n^2\left(\omega+\kappa^2\alpha\right)V^2+4n^2\beta V^3+4n^2\gamma V^4=0\end{aligned} \tag{102}$$

We will now analyze Eq. (102) to construct soliton solutions by generalized Kudryashov method. We substitute Eqs. (40), (42) and (43) into Eq. (102). Then, we employ the balance principle and determine a relation of M and N as

$$N=M+1 \tag{103}$$

Case-1: When $M=0$ and $N=1$ in Eq. (103), then we have the solution of Eq. (102) in the form

$$V(\xi)=\frac{k_0+k_1Q(\xi)}{l_0} \tag{104}$$

where $k_1\neq 0$ and $l_0\neq 0$. Substituting Eq. (104) into Eq. (102), we obtain a system of algebraic equations. Solving this system, we find the following results:

Set-1.

$$\begin{aligned}&k_0=0,\quad k_1=\frac{l_0(1+n)(d+\alpha)}{2n^2\beta},\quad l_0=l_0,\\&\gamma=-\frac{n^2\beta^2(1+2n)}{(1+n)^2(d+\alpha)},\quad \omega=-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\end{aligned} \tag{105}$$

Set-2.

$$\begin{aligned}&k_0=\frac{l_0(1+n)(d+\alpha)}{2n^2\beta},\quad k_1=-\frac{l_0(1+n)(d+\alpha)}{2n^2\beta},\\&l_0=l_0,\quad \gamma=-\frac{n^2\beta^2(1+2n)}{(1+n)^2(d+\alpha)},\quad \omega=-\alpha\kappa^2+\frac{d+\alpha}{4n^2},\end{aligned} \tag{106}$$

where κ is an arbitrary constant. Substituting Eqs. (105)-(106) along with $Q(\xi)=1/(1\pm e^{\xi})$ into Eq. (104) and using the transformation $U=V^{\frac{1}{2n}}$, we obtain exact solution to Eq. (101). Consequently, we have the exact 1-soliton solutions to Eq. (99) as follows:

By using the results in Eq. (105), we have

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1-\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \tag{107}$$

and

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1-\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (108)$$

By using the results in Eq. (106), we obtain

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1+\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (109)$$

and

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1+\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (110)$$

Case-2: When $M=1$ and $N=2$ in Eq. (103), then we have the solution of Eq. (102) in the form

$$V(\xi)=\frac{k_0+k_1Q(\xi)+k_2Q^2(\xi)}{l_0+l_1Q(\xi)} \quad (111)$$

where $k_2\neq 0$ and $l_1\neq 0$. Substituting Eq. (111) into Eq. (102), we obtain a system of algebraic equations. Solving this system, we find the following results:

Set-1.

$$k_0=0,\quad k_1=\frac{l_0(1+n)(d+\alpha)}{2n^2\beta},$$
$$k_2=\frac{l_1(1+n)(d+\alpha)}{2n^2\beta}, l_0=l_0,\quad l_1=l_1, \quad (112)$$
$$\gamma=-\frac{n^2\beta^2(1+2n)}{(1+n)^2(d+\alpha)},\quad \omega=-\alpha\kappa^2+\frac{d+\alpha}{4n^2}$$

Set-2.

$$k_0=0,\quad k_1=\frac{(2l_0+l_1)(1+n)(d+\alpha)}{2n^2\beta},$$
$$k_2=-\frac{(2l_0+l_1)(1+n)(d+\alpha)}{2n^2\beta},\quad l_0=l_0,\quad l_1=l_1, \quad (113)$$
$$\gamma=-\frac{l_1^2n^2\beta^2(1+2n)}{(2l_0+l_1)^2(1+n)^2(d+\alpha)},\quad \omega=-\alpha\kappa^2+\frac{d+\alpha}{4n^2}$$

Set-3.

$$k_0=\frac{l_0(1+n)(d+\alpha)}{2n^2\beta},\quad k_1=-\frac{(l_0-l_1)(1+n)(d+\alpha)}{2n^2\beta},$$
$$k_2=-\frac{l_1(1+n)(d+\alpha)}{2n^2\beta},\quad l_0=l_0,\quad l_1=l_1, \quad (114)$$
$$\gamma=-\frac{n^2\beta^2(1+2n)}{(1+n)^2(d+\alpha)},\quad \omega=-\alpha\kappa^2+\frac{d+\alpha}{4n^2}$$

where κ is an arbitrary constant. Consequently, we obtain the following exact traveling wave solutions to the R-NLSE with dual-power law nonlinearity:

By using the results in Eq. (112), we find exact 1-soliton solutions as

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1-\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (115)$$

and

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1-\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (116)$$

By using the results in Eq. (113), we have solitary wave solutions as

$$\psi(x,t)=\left\{\frac{(2l_0+l_1)(1+n)(d+\alpha)sech^2\left[\frac{x+2\alpha\kappa t}{2}\right]}{4n^2\beta\left(2l_0+l_1-l_1\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (117)$$

and

$$\psi(x,t)=\left\{-\frac{(2l_0+l_1)(1+n)(d+\alpha)csch^2\left[\frac{x+2\alpha\kappa t}{2}\right]}{4n^2\beta\left(2l_0+l_1-l_1\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)}\right\}^{\frac{1}{2n}} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (118)$$

By using the results in Eq. (114), we find exact 1-soliton solutions as

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1+\tanh\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (119)$$

and

$$\psi(x,t)=\left\{\frac{(1+n)(d+\alpha)}{4n^2\beta}\left(1+\coth\left[\frac{x+2\alpha\kappa t}{2}\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{d+\alpha}{4n^2}\right)t+\theta\right\}} \quad (120)$$

5. Extended trial equation method

In this section, we describe the extended trial equation method [20], [29] for finding traveling wave solutions of nonlinear partial differential equations (NLPDE) and subsequently will apply this method to solve the R-NLSE. We suppose that the given NLPDE for $u(x,t)$ is in the form

$$P\left(u,u_t,u_x,u_{xx},u_{xt},u_{tt},...\right)=0 \quad (121)$$

where P is a polynomial. The essence of the extended trial equation method can be presented in the following steps:

Step-1: To find the traveling wave solutions of Eq. (121), we introduce the wave variable

$$u(x,t)=U(\xi),\ \ \xi=x-vt \quad (122)$$

where v is a constant to be determined later. Substituting Eq. (122) into Eq. (121), we obtain the following ODE

$$Q\left(U,U',U'',...\right)=0 \quad (123)$$

Step-2: Take transformation and trial equation as follows:

$$U=\sum_{i=0}^{\varsigma}\tau_i\Psi^i \quad (124)$$

where

$$(\Psi')^2=\Lambda(\Psi)=\frac{\Phi(\Psi)}{\Upsilon(\Psi)}=\frac{\mu_\sigma\Psi^\sigma+\cdots+\mu_1\Psi+\mu_0}{\chi_\rho\Psi^\rho+\cdots+\chi_1\Psi+\chi_0} \quad (125)$$

Using the relations (124) and (125), we can find

$$(U')^2=\frac{\Phi(\Psi)}{\Upsilon(\Psi)}\left(\sum_{i=0}^{\varsigma}i\tau_i\Psi^{i-1}\right)^2 \quad (126)$$

$$U''=\frac{\Phi'(\Psi)\Upsilon(\Psi)-\Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)}\left(\sum_{i=0}^{\varsigma}i\tau_i\Psi^{i-1}\right)+\frac{\Phi(\Psi)}{\Upsilon(\Psi)}\left(\sum_{i=0}^{\varsigma}i(i-1)\tau_i\Psi^{i-2}\right) \quad (127)$$

where $\Phi(\Psi)$ and $\Upsilon(\Psi)$ are polynomials. Substituting these terms into Eq. (123) yields an equation of polynomial $\Omega(\Psi)$ of Ψ:

$$\Omega(\Psi)=\rho_s\Psi^s+\cdots+\rho_1\Psi+\rho_0=0 \quad (128)$$

According to the balance principle we can determine a relation of ρ, σ, and ς. We can take some values of ρ, σ, and ς.

Step-3: Let the coefficients of $\Omega(\Psi)$ all be zero will yield an algebraic equations system:

$$\rho_i=0,\quad i=0,...,s \quad (129)$$

Solving this equations system (129), we will determine the values of $\chi_0,...,\chi_\rho$; $\mu_0,...,\mu_\sigma$ and $\tau_0,...,\tau_\varsigma$.

Step-4: Reduce Eq. (125) to the elementary integral form,

$$\pm(\xi-\xi_0)=\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}}=\int\sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}}d\Psi \quad (130)$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Psi)$, we solve the infinite integral (130) and obtain the exact solutions to Eq. (123). Furthermore, we can write the exact traveling wave solutions to Eq. (121) respectively.

5.1. Application to R-NLSE (Kerr law)

We will now analyze Eq. (45) to obtain soliton solutions by extended trial equation method. We substitute Eqs. (124) and (127) into Eq. (47). Then, we use the balance principle and find that

$$\sigma=\rho+2\varsigma+2 \quad (131)$$

When $\sigma=4$, $\rho=0$ and $\varsigma=1$ in Eq. (131), we obtain

$$U=\tau_0+\tau_1\Psi \quad (132)$$

$$U''=\frac{\tau_1(4\mu_4\Psi^3+3\mu_3\Psi^2+2\mu_2\Psi+\mu_1)}{2\chi_0} \quad (133)$$

where $\mu_4\neq0$, $\chi_0\neq0$. Substituting Eqs. (132) and (133) into Eq. (47), collecting the coefficients of Ψ, and solving the resulting system, we find the following results:

$$\mu_2=\frac{\mu_1\tau_1}{2\tau_0}-\frac{2\beta\tau_0^2\chi_0}{\alpha+\gamma},\quad \mu_3=-\frac{2\beta\tau_0\tau_1\chi_0}{\alpha+\gamma},$$
$$\mu_4=-\frac{\beta\tau_1^2\chi_0}{2(\alpha+\gamma)},\quad \mu_0=\mu_0,\quad \mu_1=\mu_1,\quad \chi_0=\chi_0, \quad (134)$$
$$\tau_0=\tau_0,\quad \tau_1=\tau_1,\quad \omega=-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}$$

Substituting these results into Eqs. (125) and (130), we find that

$$\pm(\xi-\xi_0)=W\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}} \tag{135}$$

where

$$\Lambda(\Psi)=\Psi^4+\frac{\mu_3}{\mu_4}\Psi^3+\frac{\mu_2}{\mu_4}\Psi^2+\frac{\mu_1}{\mu_4}\Psi+\frac{\mu_0}{\mu_4},\quad W=\sqrt{\frac{\chi_0}{\mu_4}} \tag{136}$$

Integrating Eq. (135), and inserting the result into Eq. (132), then we obtain the exact solutions to Eq. (47). Consequently, we achieve the traveling wave solutions to the R-NLSE with Kerr law nonlinearity (45) as the following:

When $\Lambda(\Psi)=(\Psi-\lambda_1)^4$, then we obtain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1\pm\frac{\tau_1 W}{x+2\alpha\kappa t-\xi_0}\right\}\cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \tag{137}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$, then we get

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(x+2\alpha\kappa t-\xi_0\right)\right]^2}\right\}\cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \tag{138}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$, then we have

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{(\lambda_2-\lambda_1)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(x+2\alpha\kappa t-\xi_0\right)\right]-1}\right\}\cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \tag{139}$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(x+2\alpha\kappa t-\xi_0\right)\right]-1}\right\}\cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \tag{140}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, then we attain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2\lambda\tau_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{\lambda}}{W}\left[x+2\alpha\kappa t\right]\right)}\right\}\cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}}$$

$$\lambda=(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) \tag{141}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, then we achieve

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)sn^2\left[\pm\frac{\sqrt{\Lambda}}{2W}\left[x+2\alpha\kappa t-\xi_0\right],l\right]}\right\}\cdot\times e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}},$$

$$\Lambda=(\lambda_1-\lambda_3)(\lambda_2-\lambda_4) \tag{142}$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}$$

Also, λ_i $(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0 \tag{143}$$

When $\tau_0=-\tau_1\lambda_1$ and $\xi_0=0$, then we can reduce the solutions (137)-(141) to plane wave solutions

$$q(x,t)=\left\{\pm\frac{\tau_1 W}{x+2\alpha\kappa t}\right\}e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \tag{144}$$

$$q(x,t)=\left\{\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(x+2\alpha\kappa t\right)\right]^2}\right\}\cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \tag{145}$$

singular soliton solutions

$$q(x,t)=\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}\left(x+2\alpha\kappa t\right)\right]\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \quad (146)$$

and bright soliton solution

$$q(x,t)=\left\{\frac{A}{D+\cosh[B(x+2\alpha\kappa t)]}\right\} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \quad (147)$$

where

$$A=\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2},\quad B=\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W},\quad D=\frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2} \quad (148)$$

Here, A is the amplitude of the soliton, while B is the inverse width of the soliton. These solitons exist for $\tau_1<0$. Moreover, when $\tau_0=-\tau_1\lambda_2$ and $\xi_0=0$, we can write the Jacobi elliptic function solution (142) as

$$q(x,t)=\left\{\frac{A_1}{D_1+sn^2\left[B_j\left[x+2\alpha\kappa t\right],\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]}\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \quad (149)$$

where

$$A_1=\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4},\quad D_1=\frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4},\quad B_j=\frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W},\ (j=1,2) \quad (150)$$

Remark-1: When the modulus $l\to 1$, we can reduce the solution (149) to a second form of singular optical soliton solutions as

$$q(x,t)=\left\{\frac{A_1}{D_1+\tanh^2\left[B_j\left(x+2\alpha\kappa t\right)\right]}\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \quad (151)$$

where $\lambda_3=\lambda_4$.

Remark-2: However, if $l\to 0$, we can reduce the solution (149) to the periodic singular solutions as

$$q(x,t)=\left\{\frac{A_1}{D_1+\sin^2\left[B_j\left(x+2\alpha\kappa t\right)\right]}\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\beta\tau_0^2+\frac{\mu_1\tau_1(\alpha+\gamma)}{2\tau_0\chi_0}\right)t+\theta\right\}} \quad (152)$$

where $\lambda_2=\lambda_3$.

5.2. Application to R-NLSE (Power law)

We will now analyze Eq. (62) to construct soliton solutions by extended trial equation method. We substitute Eqs. (124), (126) and (127) into Eq. (65). Then, we use the balance principle and find that

$$\sigma=\rho+\varsigma+2 \quad (153)$$

Case-1: When $\sigma=3$, $\rho=0$ and $\varsigma=1$ in Eq. (153), we have

$$V=\tau_0+\tau_1\Psi \quad (154)$$

$$(V')^2=\frac{\tau_1^2(\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0} \quad (155)$$

$$V''=\frac{\tau_1(3\mu_3\Psi^2+2\mu_2\Psi+\mu_1)}{2\chi_0} \quad (156)$$

where $\mu_3\neq 0$, $\chi_0\neq 0$. Substituting Eqs. (154)-(156) into Eq. (65), and solving the resulting system of algebraic equations, we find the following results:

$$\begin{aligned}
\mu_0&=\frac{\tau_0^2\left[\mu_2(1+n)(\alpha+\gamma)+8n^2\beta\tau_0\chi_0\right]}{\tau_1^2(1+n)(\alpha+\gamma)},\\
\mu_1&=\frac{2\tau_0\left[\mu_2(1+n)(\alpha+\gamma)+6n^2\beta\tau_0\chi_0\right]}{\tau_1(1+n)(\alpha+\gamma)},\\
\mu_3&=-\frac{4n^2\beta\tau_1\chi_0}{(1+n)(\alpha+\gamma)},\\
\omega&=-\alpha\kappa^2+\frac{3\beta\tau_0}{1+n}+\frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0},\\
\mu_2&=\mu_2,\quad \chi_0=\chi_0,\quad \tau_0=\tau_0,\quad \tau_1=\tau_1
\end{aligned} \quad (157)$$

Substituting these results into Eqs. (125) and (130), we find that

$$\pm(\xi-\xi_0)=\sqrt{W_1}\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}} \quad (158)$$

where

$$\Lambda(\Psi)=\Psi^3+\frac{\mu_2}{\mu_3}\Psi^2+\frac{\mu_1}{\mu_3}\Psi+\frac{\mu_0}{\mu_3},\quad W_1=\frac{\chi_0}{\mu_3} \quad (159)$$

Integrating Eq. (158), and inserting the result into Eq. (154), and using the transformation $U = V^{\frac{1}{2n}}$, then we attain the exact solutions to Eq. (64). Consequently, we write the traveling wave solutions to the R-NLSE with power law nonlinearity (62) as the following:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3$, then we obtain rational function solution as follows:

$$q(x,t) = \left\{ \tau_0 + \tau_1\lambda_1 + \frac{4\tau_1 W_1}{\left[x + 2\alpha\kappa t - \xi_0\right]^2} \right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{160}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, then we have solitary wave solution as follows:

$$q(x,t) = \left\{ \begin{aligned} &\tau_0 + \tau_1\lambda_2 \\ &+\tau_1(\lambda_1 - \lambda_2)\tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}\left[x + 2\alpha\kappa t - \xi_0\right]\right) \end{aligned} \right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{161}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)^2$ and $\lambda_1 > \lambda_2$, then we attain hyperbolic function solution as follows:

$$q(x,t) = \left\{ \begin{aligned} &\tau_0 + \tau_1\lambda_1 \\ &+\tau_1(\lambda_1 - \lambda_2)\text{cosech}^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}\left[x + 2\alpha\kappa t\right]\right) \end{aligned} \right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{162}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, then we have Jacobi elliptic function solutions as follows:

$$q(x,t) = \left\{ \begin{aligned} &\tau_0 + \tau_1\lambda_3 \\ &+\tau_1(\lambda_2 - \lambda_3)sn^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_1}}\left[x + 2\alpha\kappa t - \xi_0\right], l\right) \end{aligned} \right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{163}$$

where

$$l^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \tag{164}$$

Also, λ_i $(i = 1,2,3)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0 \tag{165}$$

When $\tau_0 = -\tau_1\lambda_1$ and $\xi_0 = 0$, then we can reduce the solutions (160)-(162) to rational function solution

$$q(x,t) = \left\{ \frac{A}{x + 2\alpha\kappa t} \right\}^{\frac{1}{n}} e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{166}$$

1-soliton solution

$$q(x,t) = \left\{ \frac{A_2}{\cosh^{\frac{1}{n}}\left[B_3\left(x + 2\alpha\kappa t\right)\right]} \right\} e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{167}$$

and singular soliton solution

$$q(x,t) = \left\{ \frac{A_3}{\sinh^{\frac{1}{n}}\left[B_3\left(x + 2\alpha\kappa t\right)\right]} \right\} e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{168}$$

where

$$\begin{aligned} &A = 2\sqrt{\tau_1 W_1}, \quad A_2 = [\tau_1(\lambda_2 - \lambda_1)]^{\frac{1}{2n}}, \\ &A_3 = [\tau_1(\lambda_1 - \lambda_2)]^{\frac{1}{2n}}, \quad B_3 = \frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}} \end{aligned} \tag{169}$$

Here, A_2 and A_3 are respectively the amplitudes of 1-soliton and singular soliton, while B_3 is the inverse width of the solitons. These solitons exist for $\tau_1 > 0$. Moreover, when $\tau_0 = -\tau_1\lambda_3$ and $\xi_0 = 0$, we can simplify the Jacobi elliptic function solution (163) as follows:

$$q(x,t) = A_4 sn^{\frac{1}{n}}\left(B_j\left[x + 2\alpha\kappa t\right], \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}\right) \cdot e^{i\left\{-\kappa x + \left(-\alpha\kappa^2 + \frac{3\beta\tau_0}{1+n} + \frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t + \theta\right\}} \tag{170}$$

where

$$A_4 = [\tau_1(\lambda_2 - \lambda_3)]^{\frac{1}{2n}}, \quad B_j = \frac{(-1)^j}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_1}}, \ (j = 4,5) \tag{171}$$

Remark-3: When the modulus $l \to 1$, dark soliton solutions fall out:

$$q(x,t)=A_4\tanh^{\frac{1}{n}}\left[B_j\left(x+2\alpha\kappa t\right)\right]\cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{3\beta\tau_0}{1+n}+\frac{\mu_2(\alpha+\gamma)}{4n^2\chi_0}\right)t+\theta\right\}} \quad (172)$$

where $\lambda_1=\lambda_2$.

Case-2: When $\sigma=4$, $\rho=0$ and $\varsigma=2$ in Eq. (153), we have

$$V=\tau_0+\tau_1\Psi+\tau_2\Psi^2 \quad (173)$$

$$(V')^2=\frac{(\tau_1+2\tau_2\Psi)^2(\mu_4\Psi^4+\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0} \quad (174)$$

$$V''=\frac{(\tau_1+2\tau_2\Psi)(4\mu_4\Psi^3+3\mu_3\Psi^2+2\mu_2\Psi+\mu_1)}{2\chi_0}+\frac{2\tau_2(\mu_4\Psi^4+\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0} \quad (175)$$

where $\mu_4\neq 0$, $\chi_0\neq 0$. Substituting Eqs. (173)-(175) into Eq. (65), and solving the resulting system of algebraic equations, we find the following results:

$$\mu_0=-\frac{n^2\beta\tau_0^2\chi_0}{\tau_2(1+n)(\alpha+\gamma)},\quad \mu_1=-\frac{2n^2\beta\tau_0\tau_1\chi_0}{\tau_2(1+n)(\alpha+\gamma)},$$
$$\mu_2=-\frac{n^2\beta\chi_0\left(\tau_1^2+2\tau_0\tau_2\right)}{\tau_2(1+n)(\alpha+\gamma)},\quad \mu_3=-\frac{2n^2\beta\tau_1\chi_0}{(1+n)(\alpha+\gamma)},$$
$$\mu_4=-\frac{n^2\beta\tau_2\chi_0}{(1+n)(\alpha+\gamma)}, \quad (176)$$
$$\omega=-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)},$$
$$\chi_0=\chi_0,\quad \tau_0=\tau_0,\quad \tau_1=\tau_1,\quad \tau_2=\tau_2$$

Substituting these results into Eqs. (125) and (130), we find that

$$\pm(\xi-\xi_0)=W_2\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}} \quad (177)$$

where

$$\Lambda(\Psi)=\Psi^4+\frac{\mu_3}{\mu_4}\Psi^3+\frac{\mu_2}{\mu_4}\Psi^2+\frac{\mu_1}{\mu_4}\Psi+\frac{\mu_0}{\mu_4},\quad W_2=\sqrt{\frac{\chi_0}{\mu_4}} \quad (178)$$

Integrating Eq. (177) and taking $\xi_0=0$, then we have the traveling wave solutions to the R-NLSE with power law nonlinearity (62) as the following:

When $\Lambda(\Psi)=(\Psi-\lambda_1)^4$, then we obtain

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\lambda_1\pm\frac{W_2}{x+2\alpha\kappa t}\right)^i\right]^{\frac{1}{2n}}\cdot e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}} \quad (179)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$, then we get

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\lambda_1+\frac{4W_2^2(\lambda_2-\lambda_1)}{4W_2^2-\left[(\lambda_1-\lambda_2)\left(x+2\alpha\kappa t\right)\right]^2}\right)^i\right]^{\frac{1}{2n}}\cdot e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}} \quad (180)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$, then we have

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\lambda_2+\frac{\lambda_2-\lambda_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_2}\left(x+2\alpha\kappa t\right)\right]-1}\right)^i\right]^{\frac{1}{2n}}\cdot e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}} \quad (181)$$

and

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\lambda_1+\frac{\lambda_1-\lambda_2}{\exp\left[\frac{\lambda_1-\lambda_2}{W_2}\left(x+2\alpha\kappa t\right)\right]-1}\right)^i\right]^{\frac{1}{2n}}\cdot e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}} \quad (182)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, then we attain

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\Omega^i\right]^{\frac{1}{2n}}e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}}.$$
$$\Omega=\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_2}\left(x+2\alpha\kappa t\right)\right]} \quad (183)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, then we achieve

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\Omega_1^{\,i}\right]^{\frac{1}{2n}}e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}},$$

$$\Omega_1=\lambda_2+\frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)sn^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_2}\left(x+2\alpha\kappa t\right),l\right]} \tag{184}$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)} \tag{185}$$

Also, λ_i $(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0 \tag{186}$$

Remark-4: When the modulus $l\to 1$, the hyperbolic function solutions fall out:

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\Omega_2^{\,i}\right]^{\frac{1}{2n}}e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}},$$

$$\Omega_2=\lambda_2+\frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\tanh^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_2}\left(x+2\alpha\kappa t\right)\right]} \tag{187}$$

where $\lambda_3=\lambda_4$.

Remark-5: However, if $l\to 0$, the periodic wave solution are listed as follows:

$$q(x,t)=\left[\sum_{i=0}^{2}\tau_i\Omega_3^{\,i}\right]^{\frac{1}{2n}}e^{i\left\{-\kappa x+\left(-\frac{\beta\tau_1^2+4\tau_2\left[\alpha\kappa^2(1+n)-\beta\tau_0\right]}{4\tau_2(1+n)}\right)t+\theta\right\}},$$

$$\Omega_3=\lambda_2+\frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\sin^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_2}\left(x+2\alpha\kappa t\right)\right]} \tag{188}$$

where $\lambda_2=\lambda_3$.

5.3. Application to R-NLSE (Parabolic law)

We will now analyze Eq. (77) to construct soliton solutions by extended trial equation method. We substitute Eqs. (124), (126) and (127) into Eq. (80). Then, we use the balance principle and find that

$$\sigma=\rho+2\varsigma+2 \tag{189}$$

When $\sigma=4$, $\rho=0$ and $\varsigma=1$ in Eq. (189), we have

$$V=\tau_0+\tau_1\Psi \tag{190}$$

$$(V')^2=\frac{\tau_1^2(\mu_4\Psi^4+\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0} \tag{191}$$

$$V''=\frac{\tau_1(4\mu_4\Psi^3+3\mu_3\Psi^2+2\mu_2\Psi+\mu_1)}{2\chi_0} \tag{192}$$

where $\mu_4\neq 0$, $\chi_0\neq 0$. Substituting Eqs. (190)-(192) into Eq. (80), collecting the coefficients of Ψ, and solving the resulting system, we find the following results:

$$\begin{aligned}&\mu_1=\frac{(d+\alpha)\left(\mu_2\tau_0^2+\mu_0\tau_1^2\right)+2\tau_0^3\chi_0\left(\beta+2\gamma\tau_0\right)}{\tau_0\tau_1(d+\alpha)},\\ &\mu_3=-\frac{2\tau_1\chi_0\left(3\beta+8\gamma\tau_0\right)}{3(d+\alpha)},\ \mu_4=-\frac{4\gamma\tau_1^2\chi_0}{3(d+\alpha)},\\ &\omega=\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right),\\ &\mu_0=\mu_0,\quad \mu_2=\mu_2,\quad \chi_0=\chi_0,\quad \tau_0=\tau_0,\quad \tau_1=\tau_1\end{aligned} \tag{193}$$

Substituting these results into Eqs. (125) and (130), we find that

$$\pm(\xi-\xi_0)=W_3\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}} \tag{194}$$

where

$$\Lambda(\Psi)=\Psi^4+\frac{\mu_3}{\mu_4}\Psi^3+\frac{\mu_2}{\mu_4}\Psi^2+\frac{\mu_1}{\mu_4}\Psi+\frac{\mu_0}{\mu_4},\quad W_3=\sqrt{\frac{\chi_0}{\mu_4}} \tag{195}$$

Integrating Eq. (194), and inserting the result into Eq. (190), and using the transformation $U=V^{\frac{1}{2}}$ then we get the exact solutions to Eq. (79). Consequently, we obtain the traveling wave solutions to the R-NLSE with parabolic law nonlinearity (77) as the following:

When $\Lambda(\Psi)=(\Psi-\lambda_1)^4$, then we obtain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1\pm\frac{\tau_1 W_3}{x+2\alpha\kappa t-\xi_0}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{196}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$, then we get

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{4W_3^2(\lambda_2-\lambda_1)\tau_1}{4W_3^2-\left[(\lambda_1-\lambda_2)\left(x+2\alpha\kappa t-\xi_0\right)\right]^2}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{197}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$, then we have

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{(\lambda_2-\lambda_1)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}\left(x+2\alpha\kappa t-\xi_0\right)\right]-1}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{198}$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}\left(x+2\alpha\kappa t-\xi_0\right)\right]-1}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{199}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, then we attain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_3}\left(x+2\alpha\kappa t\right)\right]}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{200}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, then we achieve

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)sn^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_3}\left[x+2\alpha\kappa t-\xi_0\right],l\right]}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{201}$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)} \tag{202}$$

Also, λ_i $(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0 \tag{203}$$

When $\tau_0=-\tau_1\lambda_1$ and $\xi_0=0$, then we can reduce the solutions (196)-(200) to plane wave solutions

$$q(x,t)=\left\{\pm\frac{\tau_1 W_3}{x+2\alpha\kappa t}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{204}$$

$$q(x,t)=\left\{\frac{4W_3^2(\lambda_2-\lambda_1)\tau_1}{4W_3^2-\left[(\lambda_1-\lambda_2)\left(x+2\alpha\kappa t\right)\right]^2}\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{205}$$

singular soliton solutions

$$q(x,t)=\left\{\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left(1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W_3}\left(x+2\alpha\kappa t\right)\right]\right)\right\}^{\frac{1}{2}}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{206}$$

and bright soliton solutions

$$q(x,t)=\left\{\frac{A_5}{\left(D_2+\cosh[B_6\left(x+2\alpha\kappa t\right)]\right)^{\frac{1}{2}}}\right\}\cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{207}$$

where

$$A_5 = \left(\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2}\right)^{\frac{1}{2}},$$
$$B_6 = \frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_3}, \tag{208}$$
$$D_2 = \frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}$$

Here, A_5 is the amplitude of the soliton, while B_6 is the inverse width of the soliton. These solitons exist for $\tau_1 < 0$. Moreover, when $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, we can write the Jacobi elliptic function solution (201) as

$$q(x,t) = \left\{ \frac{A_6}{\left(D_3 + sn^2\left[B_j\left(x+2\alpha\kappa t\right), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{\frac{1}{2}}} \right\} \cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{209}$$

where

$$A_6 = \left(\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{\frac{1}{2}}, \quad D_3 = \frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4},$$
$$B_j = \frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_3}, \; (j=7,8) \tag{210}$$

Remark-6: When the modulus $l \to 1$, a second form of singular optical soliton solutions fall out:

$$q(x,t) = \left\{ \frac{A_6}{\left(D_3 + \tanh^2\left[B_j\left(x+2\alpha\kappa t\right)\right]\right)^{\frac{1}{2}}} \right\} \cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{211}$$

where $\lambda_3 = \lambda_4$.

Remark-7: However, if $l \to 0$, periodic singular solutions are listed as follows:

$$q(x,t) = \left\{ \frac{A_6}{\left(D_3 + \sin^2\left[B_j\left(x+2\alpha\kappa t\right)\right]\right)^{\frac{1}{2}}} \right\} \cdot e^{i\left\{-\kappa x+\frac{1}{4}\left(-4\alpha\kappa^2+6\beta\tau_0+8\gamma\tau_0^2+\frac{\mu_2(d+\alpha)}{\chi_0}\right)t+\theta\right\}} \tag{212}$$

where $\lambda_2 = \lambda_3$.

5.4. Application to R-NLSE (Dual power law)

We will now analyze Eq. (99) to obtain soliton solutions by extended trial equation method. We substitute Eqs. (124), (126) and (127) into Eq. (102). Then, we use the balance principle and find that

$$\sigma = \rho + 2\varsigma + 2 \tag{213}$$

When $\sigma = 4$, $\rho = 0$ and $\varsigma = 1$ in Eq. (213), we have

$$V = \tau_0 + \tau_1\Psi \tag{214}$$

$$(V')^2 = \frac{\tau_1^2(\mu_4\Psi^4+\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0} \tag{215}$$

$$V'' = \frac{\tau_1(4\mu_4\Psi^3+3\mu_3\Psi^2+2\mu_2\Psi+\mu_1)}{2\chi_0} \tag{216}$$

where $\mu_4 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (214)-(216) into Eq. (102), collecting the coefficients of Ψ, and solving the resulting algebraic equations system, we find the following results:

$$\mu_1 = \frac{2\mu_0\tau_1}{\tau_0} - \frac{4n^2\tau_0^2\chi_0\left[\beta+2n\beta+2\gamma\tau_0(1+n)\right]}{\tau_1(1+n)(1+2n)(d+\alpha)},$$
$$\mu_2 = \frac{\mu_0\tau_1^2}{\tau_0^2} - \frac{4n^2\tau_0\chi_0\left[2\beta(1+2n)+5\gamma\tau_0(1+n)\right]}{(1+n)(1+2n)(d+\alpha)},$$
$$\mu_0 = \mu_0, \quad \mu_3 = -\frac{4n^2\tau_1\chi_0\left[\beta+2n\beta+4\gamma\tau_0(1+n)\right]}{(1+n)(1+2n)(d+\alpha)}, \tag{217}$$
$$\mu_4 = -\frac{4n^2\gamma\tau_1^2\chi_0}{(1+2n)(d+\alpha)}, \quad \chi_0 = \chi_0, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1$$
$$\omega = -\alpha\kappa^2 + \frac{\beta\tau_0}{1+n} + \frac{\gamma\tau_0^2}{1+2n} + \frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}$$

Substituting these results into Eqs. (125) and (130), we find that

$$\pm(\xi-\xi_0) = W_4\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}} \tag{218}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4},$$
$$W_4 = \sqrt{\frac{\chi_0}{\mu_4}} \tag{219}$$

Integrating Eq. (218), and inserting the result into Eq. (214), and using the transformation $U = V^{\frac{1}{2n}}$ then we attain the exact solutions to Eq. (101). Consequently, we obtain the traveling wave solutions to the R-NLSE with dual-power law nonlinearity (99) as the following:

When $\Lambda(\Psi)=(\Psi-\lambda_1)^4$, then we obtain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1\pm\frac{\tau_1 W_4}{x+2\alpha\kappa t-\xi_0}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (220)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$, then we get

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{4W_4^2(\lambda_2-\lambda_1)\tau_1}{4W_4^2-\left[(\lambda_1-\lambda_2)\left(x+2\alpha\kappa t-\xi_0\right)\right]^2}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (221)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$, then we have

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{(\lambda_2-\lambda_1)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_4}\left(x+2\alpha\kappa t-\xi_0\right)\right]-1}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (222)$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_4}\left(x+2\alpha\kappa t-\xi_0\right)\right]-1}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (223)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, then we attain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_4}\left(x+2\alpha\kappa t\right)\right]}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (224)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, then we achieve

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)sn^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_4}\left(x+2\alpha\kappa t-\xi_0\right),l\right]}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (225)$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)} \quad (226)$$

Also, λ_i $(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0 \quad (227)$$

When $\tau_0=-\tau_1\lambda_1$ and $\xi_0=0$, then we can reduce the solutions (220)-(224) to plane wave solutions

$$q(x,t)=\left\{\pm\frac{\tau_1 W_4}{x+2\alpha\kappa t}\right\}^{\frac{1}{2n}} e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (228)$$

$$q(x,t)=\left\{\frac{4W_4^2(\lambda_2-\lambda_1)\tau_1}{4W_4^2-\left[(\lambda_1-\lambda_2)\left(x+2\alpha\kappa t\right)\right]^2}\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (229)$$

singular soliton solutions

$$q(x,t)=\left\{\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left(1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W_4}\left(x+2\alpha\kappa t\right)\right]\right)\right\}^{\frac{1}{2n}} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (230)$$

and bright soliton solutions

$$q(x,t)=\left\{\frac{A_7}{\left(D_4+\cosh[B_9\left(x+2\alpha\kappa t\right)]\right)^{\frac{1}{2n}}}\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \quad (231)$$

where

$$A_7 = \left(\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2}\right)^{\frac{1}{2n}}, \quad B_9 = \frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_4},$$
$$D_4 = \frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2} \tag{232}$$

Here, A_7 is the amplitude of the soliton, while B_9 is the inverse width of the soliton. These solitons exist for $\tau_1 < 0$. Moreover, when $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, we can write the Jacobi elliptic function solution (225) as

$$q(x,t) = \left\{\frac{A_8}{\left(D_5 + sn^2\left[B_j\left(x+2\alpha\kappa t\right), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{\frac{1}{2n}}}\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \tag{233}$$

where

$$A_8 = \left(\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{\frac{1}{2n}}, \quad D_5 = \frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4},$$
$$B_j = \frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_4}, \ (j=10,11) \tag{234}$$

Remark-8: When the modulus $l \to 1$, the hyperbolic function solutions fall out:

$$q(x,t) = \left\{\frac{A_8}{\left(D_5 + \tanh^2\left[B_j\left(x+2\alpha\kappa t\right)\right]\right)^{\frac{1}{2n}}}\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \tag{235}$$

where $\lambda_3 = \lambda_4$.

Remark-9: However, if $l \to 0$, the periodic wave solutions are listed as follows:

$$q(x,t) = \left\{\frac{A_8}{\left(D_5 + \sin^2\left[B_j\left(x+2\alpha\kappa t\right)\right]\right)^{\frac{1}{2n}}}\right\} \cdot e^{i\left\{-\kappa x+\left(-\alpha\kappa^2+\frac{\beta\tau_0}{1+n}+\frac{\gamma\tau_0^2}{1+2n}+\frac{\mu_0\tau_1^2(d+\alpha)}{4n^2\tau_0^2\chi_0}\right)t+\theta\right\}} \tag{236}$$

where $\lambda_2 = \lambda_3$.

5.5. Application to R-NLSE (Log law)

In case of log law nonlinearity, there is no radiation and consequently there is no shedding of energy and is hence a preferred means of soliton communication. For log law nonlinearity,

$$F(s) = \ln s \tag{237}$$

So the resonant nonlinear Schrödinger's equation with log law nonlinearity is

$$i\psi + \alpha\psi_{xx} + \beta\ln\left(|\psi|^2\right)\psi + \gamma\left\{\frac{|\psi|_{xx}}{|\psi|}\right\}\psi = 0 \tag{238}$$

Under the travelling wave transformation

$$\psi(x,t) = U(\xi)e^{i(-\kappa x+\omega t+\theta)}, \quad \xi = x + 2\kappa\alpha t \tag{239}$$

we have

$$\left(\alpha+\gamma\right)U'' - \left(\omega+\kappa^2\alpha\right)U + 2\beta U\ln\left(U\right) = 0 \tag{240}$$

In order to obtain closed form solutions, we use the transformation

$$U = \exp\frac{1}{V} \tag{241}$$

that will reduce Eq. (240) into the ODE

$$\left(\alpha+\gamma\right)\left[(V')^2 + 2V(V')^2 - V^2V''\right] + 2\beta V^3 - (\omega+\kappa^2\alpha)V^4 = 0 \tag{242}$$

We will now analyze Eq. (242) to secure soliton solutions by extended trial equation method. We substitute Eqs. (124), (126) and (127) into Eq. (242). Then, we use the balance principle and find that

$$\sigma = \rho + \varsigma + 2 \tag{243}$$

Case-1: When $\sigma = 3$, $\rho = 0$ and $\varsigma = 1$ in Eq. (243), we have

$$V = \tau_0 + \tau_1\Psi \tag{244}$$

$$(V')^2 = \frac{\tau_1^2(\mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0} \tag{245}$$

$$V'' = \frac{\tau_1(3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0} \tag{246}$$

where $\mu_3 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (244)-(246) into Eq. (242), and solving the resulting system of algebraic equations, we find the following results:

$$\mu_0 = -\frac{2\beta\tau_0^3\chi_0}{\tau_1^2(\alpha+\gamma)}, \quad \mu_1 = -\frac{6\beta\tau_0^2\chi_0}{\tau_1(\alpha+\gamma)},$$
$$\mu_2 = -\frac{6\beta\tau_0\chi_0}{\alpha+\gamma}, \quad \mu_3 = -\frac{2\beta\tau_1\chi_0}{\alpha+\gamma}, \tag{247}$$
$$\chi_0 = \chi_0, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad \omega = -\left(\beta + \alpha\kappa^2\right)$$

Substituting these results into Eqs. (125) and (130), we find that

$$\pm(\xi - \xi_0) = \sqrt{W_5}\int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}} \tag{248}$$

where

$$\Lambda(\Psi) = \Psi^3 + \frac{\mu_2}{\mu_3}\Psi^2 + \frac{\mu_1}{\mu_3}\Psi + \frac{\mu_0}{\mu_3}, \quad W_5 = \frac{\chi_0}{\mu_3}. \tag{249}$$

Integrating Eq. (248), and inserting the result into Eqs. (241) and (244), then we attain the exact solutions to Eq. (240). Consequently, we have the exact solutions to the R-NLSE with log law non
linearity (238) as the following:
When $\Lambda(\Psi) = (\Psi - \lambda_1)^3$, then we obtain

$$q(x,t) = \exp\left[\tau_0 + \tau_1\lambda_1 + \frac{4\tau_1 W_5}{\left(x + 2\alpha\kappa t - \xi_0\right)^2}\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{250}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, then we have

$$q(x,t) = \exp\left[\tau_0 + \tau_1\lambda_2 + \tau_1(\lambda_1 - \lambda_2)\tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_5}}\left[x + 2\alpha\kappa t - \xi_0\right]\right)\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{251}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)^2$ and $\lambda_1 > \lambda_2$, then we attain

$$q(x,t) = \exp\left[\tau_0 + \tau_1\lambda_1 + \tau_1(\lambda_1 - \lambda_2)\text{cosech}^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_5}}\left[x + 2\alpha\kappa t\right]\right)\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{252}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, then we get

$$q(x,t) = \exp\left[\tau_0 + \tau_1\lambda_3 + \tau_1(\lambda_2 - \lambda_3)sn^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_5}}\left[x + 2\alpha\kappa t - \xi_0\right], l\right)\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{253}$$

where

$$l^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \tag{254}$$

Also, λ_i $(i = 1,2,3)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0 \tag{255}$$

When $\tau_0 = -\tau_1\lambda_1$ and $\xi_0 = 0$, then we can reduce the solutions (250)-(252) to the following exact solutions, respectively:

$$q(x,t) = \exp\left[\frac{\left(x + 2\alpha\kappa t\right)^2}{4\tau_1 W_5}\right] e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{256}$$

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_2 - \lambda_1)}\cosh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_5}}\left(x + 2\alpha\kappa t\right)\right)\right] \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{257}$$

and

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_1 - \lambda_2)}\sinh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_5}}\left(x + 2\alpha\kappa t\right)\right)\right] \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{258}$$

Moreover, when $\tau_0 = -\tau_1\lambda_3$ and $\xi_0 = 0$, we can simplify the exact solutions (253) as follows:

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_2 - \lambda_3)} ns^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_5}}\left[x + 2\alpha\kappa t\right], \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}\right)\right] \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{259}$$

Remark-10: When the modulus $l \to 1$, we can write the solutions (259) as

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_2 - \lambda_3)}\coth^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_5}}\left[x + 2\alpha\kappa t\right]\right)\right] \cdot e^{i\left\{-\kappa x - \left(\beta+\alpha\kappa^2\right)t + \theta\right\}} \tag{260}$$

where $\lambda_1 = \lambda_2$.

Case-2: When $\sigma = 4$, $\rho = 0$ and $\varsigma = 2$ in Eq. (243), we have

$$V = \tau_0 + \tau_1\Psi + \tau_2\Psi^2 \tag{261}$$

$$(V')^2 = \frac{(\tau_1 + 2\tau_2\Psi)^2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0} \tag{262}$$

$$V'' = \frac{(\tau_1 + 2\tau_2\Psi)(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0} + \frac{2\tau_2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0} \tag{263}$$

where $\mu_4 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (261)-(263) into Eq. (242), and solving the resulting system of algebraic equations, we find the following results:

$$\mu_0 = -\frac{\beta\tau_1^4\chi_0}{32\tau_2^3(\alpha+\gamma)}, \quad \mu_1 = -\frac{\beta\tau_1^3\chi_0}{4\tau_2^2(\alpha+\gamma)}, \quad \mu_2 = -\frac{3\beta\tau_1^2\chi_0}{4\tau_2(\alpha+\gamma)}, \quad \mu_3 = -\frac{\beta\tau_1\chi_0}{\alpha+\gamma}, \quad \mu_4 = -\frac{\beta\tau_2\chi_0}{2(\alpha+\gamma)} \quad \chi_0 = \chi_0, \quad \tau_0 = \frac{\tau_1^2}{4\tau_2}, \quad \tau_1 = \tau_1, \quad \tau_2 = \tau_2, \quad \omega = -\left(\beta + \alpha\kappa^2\right) \tag{264}$$

Substituting these results into Eqs. (125) and (130), we find that

$$\pm(\xi - \xi_0) = W_6 \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}} \tag{265}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad W_6 = \sqrt{\frac{\chi_0}{\mu_4}} \tag{266}$$

Integrating Eq. (265) and taking $\xi_0 = 0$, then we have the exact solutions to the R-NLSE with log law nonlinearity (238) as the following:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, then we obtain

$$q(x,t) = \exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_1 \pm \frac{W_6}{x + 2\alpha\kappa t}\right)^i\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta + \alpha\kappa^2\right)t + \theta\right\}} \tag{267}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, then we get

$$q(x,t) = \exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_1 + \frac{4W_6^2(\lambda_2 - \lambda_1)}{4W_6^2 - \left[(\lambda_1 - \lambda_2)\left(x + 2\alpha\kappa t\right)\right]^2}\right)^i\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta + \alpha\kappa^2\right)t + \theta\right\}} \tag{268}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, then we have

$$q(x,t) = \exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp\left[\frac{\lambda_1 - \lambda_2}{W_6}\left(x + 2\alpha\kappa t\right)\right] - 1}\right)^i\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta + \alpha\kappa^2\right)t + \theta\right\}} \tag{269}$$

and

$$q(x,t) = \exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_1 + \frac{\lambda_1 - \lambda_2}{\exp\left[\frac{\lambda_1 - \lambda_2}{W_6}\left(x + 2\alpha\kappa t\right)\right] - 1}\right)^i\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta + \alpha\kappa^2\right)t + \theta\right\}} \tag{270}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, then we attain

$$q(x,t) = \exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_1 - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh\left[\frac{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_6}\left(x + 2\alpha\kappa t\right)\right]}\right)^i\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta + \alpha\kappa^2\right)t + \theta\right\}} \tag{271}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, then we achieve

$$q(x,t) = \exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)sn^2\left[\pm\frac{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_6}\left(x + 2\alpha\kappa t\right), l\right]}\right)^i\right]^{-1} \cdot e^{i\left\{-\kappa x - \left(\beta + \alpha\kappa^2\right)t + \theta\right\}} \tag{272}$$

where

$$l^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)} \tag{273}$$

Also, λ_i $(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0 \tag{274}$$

Remark-11: When the modulus $l\to 1$, we write the solutions (272) as

$$q(x,t)=\left[\exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_2+\frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\tanh^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_6}\left(x+2\alpha\kappa t\right)\right]}\right)^i\right]\right]^{-1}\cdot e^{i\left\{-\kappa x-\left(\beta+\alpha\kappa^2\right)t+\theta\right\}} \tag{275}$$

where $\lambda_3=\lambda_4$.

Remark-12: However, if $l\to 0$, we write the solutions (272) as

$$q(x,t)=\left[\exp\left[\sum_{i=0}^{2}\tau_i\left(\lambda_2+\frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\sin^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_6}\left(x+2\alpha\kappa t\right)\right]}\right)^i\right]\right]^{-1}\cdot e^{i\left\{-\kappa x-\left(\beta+\alpha\kappa^2\right)t+\theta\right\}} \tag{276}$$

where $\lambda_2=\lambda_3$.

6. Conclusion

We used the FIM for acquiring several new exact solutions of RNLSE with power law nonlinearity and time dependent coefficients. We have acquired different types exact solutions which are rational, dark, dark-bright optical combo and new as our research from literature. It is illustrated velocity functions $w(t)$ and $v(t)$ is related with the group velocity term $a(t)$. Consequently, the FIM is crucial one to construct different types of the exact solutions of the NPDE and systems.

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