Software approaching of quantized planary dynamics of charged bosons

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In the last years, the study of quantum Hall effect (QHE) has known a consistent expansion. Later, the theory has been extended to a non-commutative background on which the quantum Hall fluid evolves and a matrix model was proposed for describing the electrons moving in the lowest Landau level of the magnetic field. Almost immediately, topics like the non-commutative QHE, or QHE in higher dimensions, in the context of classical and quantum branes or of string theories, have become main objectives of investigations. In a recent series of papers, for the first quantization level, were studied planary dynamics of a charged scalar evolving in static orthogonal magnetic and electric fields. For a complete discussion, we succeed in building an entire software package in order to reach the solutions for the Euler-Lagrange coupled field equations, in a first-order perturbative approach. Working in a relativistic draw near, for exemplification, we consider two particular cases in order to underlining the reliability of our software tool. For the second case, in our analysis, we employ the Lagrangian due to Nielsen and Olesen well-known as leading, in (2+1)-dimensions, to an infinite discrete set of soliton solutions. After deriving, in static cylindrical coordinates, the Euler-Lagrange coupled field equations, we succeed in obtaining generalized solutions of the sourceless system of equations in a first-order perturbative approach. For particular cases the obtained solutions, fit the results from the literature.

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1. Introduction

In the last years, it has been stated that the Chern – Simons theories provide a reliable support for the study of the quantum Hall effect (QHE) [1]. Soon after, the theory has been extended to a non-commutative background on which the quantum Hall fluid evolves [2] and a matrix model was projected for describing the electrons moving in the lowest Landau level of the magnetic field [3]. Soon, topics like the non-commutative QHE [4], or QHE in higher dimensions, in the context of classical and quantum branes or of string theories [5], have become main targets of researches.

Recently, working in a non-relativistic Chern–Simons theory, defined on a background whose coordinates commute, the electrons have been identified with vortices. Using the relationship among the vortices dynamics and the Maxwell and Chern–Simons theories [6] as well as their D-brane comprehension in a type IIB string theory [7], it has been shown that the vortices may form a quantum Hall fluid whose low-energy dynamics is controlled by the matrix model [8].

On the other hand, concepts developed in particle physics, as the spontaneously broken gauge symmetry and macroscopic occupation of quantum levels, have been intensively used in explaining the Meissner effect or the flux quantization in the Josephson and integer quantum Hall effects [9,10]. Furthermore, in astrophysics, the Hall effect has been considered as involved in the magnetic field evolution in neutron star shells [11] or in the star formation [12]. Recently, methods developed in particle physics have been intensively engaged in explaining the extraordinary accuracy of the basic physical laws ruling the behavior of the macroscopic systems, since they provide a surprisingly simple theoretical basis which can be experimentally verified to a elevated degree of precision [1]. In this respect, the so called topological ‘‘defects’’ which can generate a primordial magnetic field or an exotic quantum phenomena, have been employed in a wide range of theories. The simplest models that give rise to domain walls, gauge strings, magnetic monopoles and textures are based of Lagrangians of real or complex scalar fields, with a spontaneously broken symmetry [2]. On the other hand, a wide area of intensive research in mesoscopic physics has been recently developed, by considering the fundamental implications of quantum mechanics in two-dimensional samples [3]. Since the planary dynamics of charged particles evolving in static orthogonal magnetic and electric fields is of real interest—especially after it has been seen that electrons confined in two dimensions exhibit the remarkable phenomena of quantum Hall effect, a specialised software tool is welcome. It should offer a good opportunity to continue the research work, covering the huge volume of computations. In this way, basing on our previous works where we got, within a relativistic approach, the quantum eigenstates and the energy spectrum, as a non-linear dependence on the exterior fields and the particle momentum parameter, [4], we succeed in building an entire software package in order to reach the solutions for the Euler-Lagrange coupled field equations, in a first-order perturbative approach.

As an example, working in a relativistic draw near, for exemplification, were evaluated the quantum eigenstates and the energy spectrum. In the particular case of the Lagrangian due to Nielsen and Olesen, in static cilindric coordinates, was recovered the non-linear dependence of the energy-eigenvalues on the magneto-electric fields and we succeed in getting a more general expression for the gauge fields' solutions. The used platforms are MAPLE and MATLAB with symbolic and differential equations libraries.

2. Review of the software procedure sets

Lastly, in literature, in the last period of time, a series of successful symbolic computation or numerical approaches were proposed [8-13]. Such theoretical and multidisciplinary studies are important and have high applicability skills [8, 12, 13]. On the other hand, the main advantage comes from the real possibility of covering the necessary huge volumes of computations.

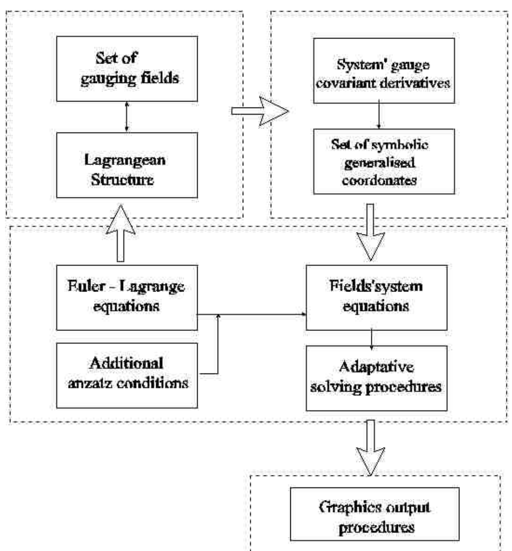

Fig.1. Modular structure of a dedicated software package

The contemporary needs and the complexity of the studied systems, lead to a structured and Object Oriented Programming approaching, in order to succeed enveloping the entire system' complexity. On the other hand, many software tools should not respond only to one particular problem. The current tendency is to design and use reliable software programs which should be able to respond to an entire series of requests. In this order, in our days, mainly are used complete software packages, which could cover all these requests. A software package is used in object-oriented programming to name a group of related classes of a program. Packages are useful to measure and control the inherent coupling of a program.

In a modular program, even outside object-oriented programming, a software package may refer to any component (module) that can be integrated into the main program. Typically this is done by the end user in a well-defined interface. A block structure of our developed application is shown in the picture below.

The first unit includes the definition procedures. It allows to describe the Lagrangian structures and to set up the coordinates' set.

In the subsequent parts, are involved fourth order recursive procedures in order to transform the considered Lagrangian structures into a convertible form due to allow the writing down the Euler Lagrange equations. In the same manner are involved specific procedures in order to succeed in solving these equation using the first-order perturbation theory.

3. Results and discussion

Let us start with the U(1)-gauge invariant Lagrangian for the charged scalar field [6,7,13],

$$\mathrm{L} = -\left(D_\mu \Phi\right)^* \left(D_\mu \Phi\right) - m_0^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (1)$$

where D_μ represents the U(1)-gauge covariant derivative, i.e.

$$D_\mu = \partial_\mu - ieA_\mu \quad (2)$$

while $F_{\mu\nu}$ is the Maxwell electromagnetic tensor.

The corresponding Euler–Lagrange scalar field equation

$$\partial_\mu \left(\frac{\partial \mathrm{L}}{\partial \Phi_{;\mu}} \right) - \frac{\partial \mathrm{L}}{\partial \Phi} = 0 \qquad \text{and its h.c.} \quad (3)$$

could be written as

$$\begin{aligned} &\partial_\mu \partial_\mu \Phi(r,t) - 2ie\vec{A}\vec{\nabla}\Phi(r,t) - 2ieV\frac{\partial \Phi(r,t)}{\partial t} - \\ &-\left[m_0^2 + e^2\left(\vec{A}^2 - V^2\right)\right]\Phi(r,t) = 0 \end{aligned} \quad (4)$$

In order to succeed in building a complete comparison with the previous achieved results from the literature [5, 6, 7, 13], we should include the supplementary Assatz conditions,

$$A_x = A_z = 0 \quad (5)$$

$$A_y = B_0 x \quad (6)$$

$$V = E_0 x \quad (7)$$

where E_0 and B_0 are the orthogonal electric and magnetic fields.

The explicit form of the scalar field equation becomes

$$\mu^{ij}\Phi_{,ij} - 2ieB_0 x\Phi_{,y} + 2ieE_0 x\Phi_{,t} - \\ -\left[m_0^2 + e^2 x^2\left(B_0^2 - E_0^2\right)\right]\Phi = 0 \tag{8}$$

and its h.c., where μ^{ij} is the Minkowski metric tensor, i.e.

$$\mu^{ij} = diag[\ \ 1,\ \ 1,\ \ 1,\ -1]$$

Using the Compton recalibration,

$$\xi = m_0 x \tag{9}$$

$$\eta = \mu_0 y \tag{10}$$

the field equation (8) becomes

$$m_0^2\left(\frac{\partial^2\Phi(\xi,\eta,t)}{\partial\xi^2}\right) + m_0^2\left(\frac{\partial^2\Phi(\xi,\eta,t)}{\partial\eta^2}\right) - \left(\frac{\partial^2\Phi(\xi,\eta,t)}{\partial t^2}\right) - \\ -2ei\left[B_0\xi\left(\frac{\partial\Phi(\xi,\eta,t)}{\partial\eta}\right) + \frac{E_0\xi}{\eta}\left(\frac{\partial\Phi(\xi,\eta,t)}{\partial t}\right)\right] - \\ -\left\{m_0^2 + e^2\left[\left(\frac{B_0\xi}{m_0}\right)^2 - \left(\frac{E_0\xi}{m_0}\right)^2\right]\right\}\Phi(x,t) = 0$$

or, in an more compact form, could be written as

$$m_0^2\left(\Phi_{,\xi\xi} + \Phi_{,\eta\eta}\right) - \Phi_{,tt} - \\ -2eiB_0\xi\Phi_{,\eta} - 2ei\frac{E_0\xi}{m_0}\Phi_{,t} - \\ -\left\{m_0^2 + e^2\left[\left(\frac{B_0\xi}{m_0^2}\right)^2 - \left(\frac{E_0\xi}{m_0^2}\right)^2\right]\right\}\Phi = 0 \tag{11}$$

and its h.c.

These equations admit variables separation (1,13) and performing it by considering

$$\Phi(\xi,\eta,t) = \chi(\xi)\exp[i\kappa\eta]\exp[-i\omega t] \tag{12}$$

we arrive at the expression (11) at the form

$$\frac{d^2\chi}{d\xi^2} + \left[\frac{\omega^2}{m_0^2} - \kappa^2 - 1\right]\chi + \\ +\left[2\left(\Omega\kappa + \alpha\frac{\omega}{m_0}\right)\xi - \left(\Omega^2 - \alpha^2\right)\xi^2\right]\chi = 0 \tag{13}$$

where we consider the definitions

$$\Omega = \frac{eB_0}{{m_0}^2} \tag{14}$$

and

$$\alpha = \frac{eE_0}{{m_0}^2} \tag{15}$$

This equation is a second order rank with linear symmetries (found with integrated *odeadvisor* help), and, considering the substitutes

$$\delta \equiv \Omega\kappa + \alpha\frac{\omega}{m_0} \quad \text{and} \quad \beta^2 \equiv \Omega^2 - \alpha^2$$

expression (13) becomes

$$\frac{d^2\chi}{d\xi^2} + \left[\frac{\omega^2}{m_0^2} - \kappa^2 - 1\right]\chi + \left[2\delta\xi - \beta^2\xi^2\right]\chi = 0 \tag{16}$$

which could be written in a more simple form, by employing the variable change

$$u = \beta\xi - \frac{\delta}{\beta} \tag{17}$$

which brings the field equation (16) at the standard form [1, 2, 5, 13] as

$$\beta^2\frac{d^2\chi}{du^2} + \left[\left(p + \frac{\delta^2}{\beta^2}\right) - u^2\right]\chi = 0 \tag{18}$$

where we used the definition

$$p = \frac{\omega^2}{m_0^2} - \kappa^2 - 1$$

The general solution for this equation could be written as

$$\chi(u) = \frac{C_1}{\sqrt{u}}WM\left(\frac{p}{4\beta}\left(1 + \frac{\delta^2}{\beta^2}\right), \frac{1}{4}, \frac{u^2}{\beta}\right) + \\ + \frac{C_2}{\sqrt{u}}WW\left(\frac{p}{4\beta}\left(1 + \frac{\delta^2}{\beta^2}\right), \frac{1}{4}, \frac{u^2}{\beta}\right) \tag{19}$$

where *WM* and *WV* are the WhittakerM and respectively WhittakerV functions [14].

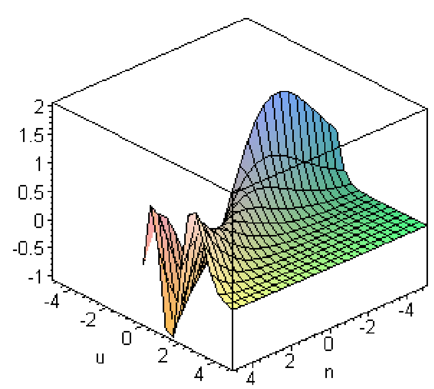

Fig.2. The general solution of the scalar field' equation (19).

This solution is more general as the previous one [13, 15]. We could recover the anterior results if the relation (19) is expressed in terms of the Hermite associated functions as for the first term of this expression, in the nearby origin domain:

$$\chi(u) = C\exp\left(-\frac{u^2}{2\beta}\right)H\left(\frac{p}{2\beta}\left(1-\beta+\frac{\delta^2}{\beta^2}\right),\frac{u^2}{\sqrt{\beta}}\right)$$

By imposing the well known condition as the model parameters must satisfy the quantization relation

$$p\beta^2 + \delta^2 = (2n+1)\beta^3 \quad n \in N \tag{20}$$

we come to an overall non-linear dependence of the energy-eigenvalues, ω_n, on the magneto-electric fields, contained in Ω and α and on the particle momentum parameter κ.

By considering E_0 to be the Hall electric field, we notice that the electromagnetic current, whose only non-vanishing component is [13,15]

$$J_y = -e\kappa C_n{}^2 \exp\left(-\frac{u^2}{\beta}\right)H_n^2\left(\frac{u}{\sqrt{\beta}}\right) \tag{21}$$

exhibits the quantization, due to the magnetic field involved in the Landau split of the stationary states.

In the second case, we made an approach for a more complex case, described by a Nielsen-Olesen Lagrangian density of the form

$$\begin{aligned} L &= (D_\mu\Phi)^*(D_\mu\Phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{\lambda}{4}\left[(\Phi^*\Phi)^2 - \eta^2\Phi^*\Phi\right] \end{aligned} \tag{22}$$

where D_μ represents the U(1)-gauge covariant derivative as above (2), $F_{\mu\nu}$ is the corresponding Maxwell tensor, defined as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu;\mu} - A_{\mu;\nu} \tag{23}$$

and $\nabla_\mu = (*)_{;\mu}$ stands for the Levi–Civita convariant derivative (as we are going to use cylindrical coordinates).

The system of Euler–Lagrange equations, namely

$$\nabla_\mu\left(\frac{\partial \mathrm{L}}{\partial \Phi^*{}_{;\mu}}\right) - \frac{\partial \mathrm{L}}{\partial \Phi^*} = 0 \qquad \text{and it's h.c.} \tag{24}$$

$$\nabla_\mu\left(\frac{\partial \mathrm{L}}{\partial A^r{}_{;\mu}}\right) - \frac{\partial \mathrm{L}}{\partial A^r} = 0 \tag{25}$$

$$\nabla_\mu\left(\frac{\partial \mathrm{L}}{\partial A^\theta{}_{;\mu}}\right) - \frac{\partial \mathrm{L}}{\partial A^\theta} = 0 \tag{26}$$

in the static case and for (r, θ)-depending fields, explicitly becomes

$$\begin{aligned} &\frac{1}{r}(r\Phi_{,r})_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta} = 2ie\left[A^r\Phi_{,r} + \frac{1}{r}A^\theta\Phi_{,\theta}\right] + \\ &+ e^2\Phi\left[\left|A^r\right|^2 + \left|A^\theta\right|^2\right] + \frac{1}{2}\lambda\Phi\left[|\Phi|^2 - \frac{1}{2}\eta^2\right] \end{aligned} \tag{27}$$

respectively

$$\begin{aligned} &A^r_{,rr} + \frac{3}{r}A^r_{,r} + \frac{1}{r^2}A^r + \frac{1}{r^2}A^r_{,\theta\theta} = \\ &= ie\left[\Phi^*\Phi_{,r} - \Phi^*_{,r}\Phi\right] + 2e^2|\Phi|^2 A^r \end{aligned} \tag{28}$$

$$\begin{aligned} &A^\theta_{,rr} + \frac{1}{r}A^\theta_{,r} + \frac{2}{r^2}A^r_{,\theta} + \frac{1}{r^2}A^\theta_{,\theta\theta} - \frac{1}{r^2}A^\theta = \\ &= \frac{ie}{r}\left[\Phi^*\Phi_{,\theta} - \Phi^*_{,\theta}\Phi\right] + 2e^2|\Phi|^2 A^\theta \end{aligned} \tag{29}$$

where the pseudo-orthonormal tetrad has been expressed in cylindrical coordinates as

$$\nabla_\mu = \left\{\partial_r, \frac{1}{r}\partial_\theta, \partial_z, \partial_t\right\} \tag{30}$$

In the same time, it has to be imposed the Coulomb condition

$$\frac{1}{r}A^r + A^r_{,r} + \frac{1}{r}A^\theta_{,\theta} = 0 \tag{31}$$

The equations' solving approach is made, as in the previous case, by considering the usual modal field decomposition of the form

$$\Phi = f(r)e^{in\theta} \tag{32}$$

and

$$A^r = v(r)g(\theta) \tag{33}$$

and respectively

$$A^\theta = \omega(r)h(\theta) \tag{34}$$

In this way, considered the Coulomb necessary condition, we arrive at the following relations [13, 15]

$$g(\theta) = e^{in\theta} \tag{35}$$

$$h(\theta) = \frac{ik}{n}e^{in\theta} \tag{36}$$

respectively

$$v(r) + rv(r)_{,r} = k\omega(r) \tag{37}$$

The radial component of the gauging field (29) becomes

$$v_{,rr}+\frac{3}{r}v_{,r}+\left[\frac{1-n^2}{r^2}-2e^2f^2\right]v=0 \quad (38)$$

Since the obtained system of coupled field equations had been required a detailed and complex numerical analysis, in a previous paper, it was delayed for the aim of a future investigation, in the end of this section we shall focus on a particular solutions of this fields' system.

Considering the first order approximation, the scalar field equation' solution admits the form

$$f(r)=C_1 BesselJ(n,m_0 r)+C_1 BesselY(n,m_0 r) \quad (39)$$

where we use the definition

$$m_0{}^2=\frac{\lambda\eta^2}{4} \quad (40)$$

This expression generalizes the previous results from the literature [14,15]. Considering the particular case for the solution written with the Bessel functions [13]

$$f(r)=J_n(m_0 r) \quad (40)$$

which is satisfied by the Bessel functions Jn(lr) [7], so that the full n-mode solution reads

$$\Phi_n(r)=J_n(\mu r)e^{in\theta} \quad (41)$$

we arrive at the general solutions in a mode-by-mode expansion read [15, 19]

$$A_r(r,\theta)=\left[\frac{C_1(n)}{r}r^n+\frac{C_2(n)}{r}r^{-n}\right]e^{in\theta} \quad (42)$$

$$A_\theta(r,\theta)=i\left[\frac{C_1(n)}{r}r^n-\frac{C_2(n)}{r}r^{-n}\right]e^{in\theta} \quad (43)$$

where the complex spectral amplitudes C_1, C_2 should satisfy the so-called reality condition for the vector potential components, i.e.

$$\overline{C}_1(-n)=C_2(n) \quad (44)$$

With these solutions could be computed the source current $j(r,\theta)$ from Klein Gordon equation (27) and the B_z nontrivial component component of the magnetic field.

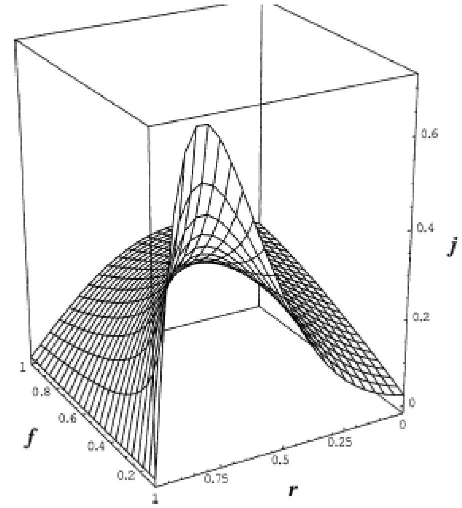

Fig.3. $j(r,\theta)$ in unit of e/m_0 for $\alpha=1$

Thence, the $\vec{B}=\nabla\times\vec{A}$ equation gets just one nontrivial component, namely:

$$B_z=\frac{\partial A_\theta}{\partial r}-\frac{1}{r}\frac{\partial A_r}{\partial\theta}+\frac{1}{r}A_\theta=$$
$$=\frac{1}{r^3}\left[r^nC(n-1)+r^{-n}\overline{C}(n+1)\right]\exp(in\theta)-$$
$$-\frac{1}{r^3}\left[r^nC+r^{-n}\overline{C}\right]ni\exp(in\theta)$$

This expression generalizes the previous results from literature [14,19].

4. Conclusions

In the present paper, is presented a coherent computer approaching strategy [17, 18] due to succeed in building, writing down and solving in the first orders of approximation, the Euler-Lagrange coupled field equations. Such computer software programs are, in the present day, very important tools for simulations and theoretical estimations. Working in a relativistic draw near, for exemplification, we consider two particular cases in order to underlining the reliability of our software tool. After deriving the Euler-Lagrange coupled field equations, we succeed in obtaining generalized solutions of the sourceless system equations in a first-order perturbative approach. All these will also help in future studies devoted to the quantized planary dynamics of charged bosons.

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