Solitons in optical metamaterials by mapping method

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This paper studies solitons in optical metamaterials by the aid of mapping method. There are two types of nonlinear media taken into consideration. They are Kerr law and parabolic law nonlinearity. The constraint conditions, on the parameters, that needs to hold for the solitons to exist, are also listed.

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1. Introduction

Optical metamaterials is a cutting edge technology that is being studied in the context of optical fibers at present times. These metamaterials carry a lot of promise and hope to address the dynamics of solitons through optical fibers. The integrability aspect of the model equation is being investigated by several authors across the globe, since its first appearance in 2011 [1-20]. Since 2011, there is always a constant thrive to extract soliton solutions to the model. There are several integration techniques that are being applied to secure soliton and other solutions to the model. Several of these results are already reported during the past few years [1-5, 11].

This paper utilizes a different and unique approach to retrieve soliton solutions to the model that is studied in metamaterials. This is the mapping method. This scheme obtains doubly periodic functions to the model and finally in the limiting case for modulus of ellipticity, solitons emerge from the mathematical analysis. The scheme is applied to two forms of nonlinear media, which are Kerr law and parabolic law. The details are described in the rest of the paper.

2. Overview of mapping method

In this section, we give an analysis of mapping methods which will be employed in this paper [6, 7, 9]. The analysis given below is in general for a system of partial differential equations (PDE)s [10] but in this paper we have applied it for a single PDE.

Consider a nonlinear coupled PDE with two dependent variables u and v and two independent variables x and t given by

$$F(u, v, u_t, v_t, u_x, v_x, u_{xxx}, v_{xxx}, ...) = 0 \tag{1}$$

where subscripts denote partial derivatives with respect to the corresponding independent variables and F is a polynomial function of the indicated variables.

Step-1: Assume that eq. (1) has a traveling wave solution in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{l_1} A_i f^i(\xi) \quad \text{(2-1)}$$

$$v(x,t) = v(\xi) = \sum_{i=0}^{l_2} B_i f^i(\xi) \quad \text{(2-2)}$$

where $\xi = x - \lambda t$, A_i , B_i and λ are arbitrary constants, l_1 and l_2 are integers and f^i represents integer powers of f . The first derivative of f with respect to ξ denoted by f' can be expressed in powers of f in the form

$$f'^2 = p\, f^2 + \frac{1}{2} q\, f^4 + r \tag{3}$$

where p , q and r are arbitrary constants. The motivation for eq. (3) was that the squares of the first derivatives of Jacobi elliptic functions (JEF)s can be expressed in even powers of themselves.

Step-2: Substituting eq. (2) into eq. (1), the PDE reduces to an ODE. Balancing the highest order derivative term and the highest order nonlinear term of the ODE, the values of l_1 and l_2 can be found.

Step-3: Substituting for u and v and using eq. (3), the ODE gives rise to a set of algebraic equations by setting the coefficients of various powers of f to zero.

Step-4: From the values of the parameters A_i, B_i, p, q and r, the solution of eq. (1) can be derived.

Thus a mapping relation is established through eq. (2) between the solution to eq. (3) and that of eq. (1). It is to be noted that if the values of l_1 and l_2 are integers, we can use the method directly to get a variety of solutions in terms of hyperbolic functions or JEFs. If they are non integers, the equation may still have solutions as rational expressions involving hyperbolic functions or JEFs.

3. Application to metamaterials

The mapping scheme described above will be applied to optical metamaterials. The governing equation for optical metamaterials is given by the nonlinear Schrödinger's equation (NLSE) given by

$$\begin{aligned} &iq_t + aq_{xx} + F(|q|^2)q = i\alpha q_x \\ &+ i\lambda(|q|^2 q)_x + i\nu(|q|^2)_x q \\ &+ \theta_1(|q|^2 q)_{xx} + \theta_2|q|^2 q_{xx} + \theta_3 q^2 q^*_{xx} = 0 \end{aligned} \tag{4}$$

In (4), the dependent variable that represents the complex valued wave profile is denoted by q and its complex conjugate is q^*. The independent variables are x and t which represents spatial and temporal variables. Next, the first term on the left hand side is linear evolution. The coefficient of a is the group velocity dispersion (GVD) and the nonlinearity is represented by the functional F. On the right hand side, α is the coefficient of intermodal dispersion, while λ is the self-steepening term to avoid the formation of shock waves and ν gives the coefficient of nonlinear dispersion. Finally the coefficients of θ_j for $j = 1, 2, 3$ are accounted for metamaterials [1-4, 11].

Also in (4), F is a real-valued algebraic function and it is necessary to have the smoothness of the complex function $F(|q|^2)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, the function $F(|q|^2)q$ is k times continuously differentiable, so that [1]

$$F(|q|^2)q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m);R^2\right) \tag{5}$$

This paper wil consider only two forms of nonlinearity. They are Kerr law and parabolic law that are discussed in details in the next two subsections.

3.1 Kerr law nonlinearity

This law arises when the refractive index of light is intensity dependent. For Kerr law nonlinearity, $F(s) = s$ and therefore this form of nonlinearity is also referred to as cubic nonlinearity. Most commercial optical fibers obey this Kerr law of nonlinearity. For Kerr law medium, the NLSE given by (4) modifies to

$$\begin{aligned} &iq_t + aq_{xx} + b|q|^2 q = i\alpha q_x \\ &+ i\lambda(|q|^2 q)_x + i\nu(|q|^2)_x q \\ &+ \theta_1(|q|^2 q)_{xx} + \theta_2|q|^2 q_{xx} + \theta_3 q^2 q^*_{xx} = 0 \end{aligned} \tag{6}$$

To derive soliton solutions, the starting hypothesis is

$$q(x,t) = P(x,t)e^{i\varphi}, \ \varphi = -\kappa x + \omega t + \theta \tag{7}$$

where κ is the wave number, ω is the soliton frequency and θ is the phase constant. Substituting eq. (7) into eq. (6) and separating them into real and imaginary parts, we obtain

$$\begin{aligned} &(\omega + \alpha\kappa + a\kappa^2)P \\ &+ [\kappa(\lambda - \kappa\theta_1 - \kappa\theta_2 - \kappa\theta_3) - b]P^3 \\ &- a\frac{\partial^2 P}{\partial^2 x} + 6P\left(\frac{\partial P}{\partial x}\right)^2 \theta_1 \\ &+ (3\theta_1 + \theta_2 + \theta_3)P^2 \frac{\partial^2 P}{\partial^2 x} = 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\frac{\partial P}{\partial t} - (\alpha + 2a\kappa)\frac{\partial P}{\partial x} = \\ &(3\lambda + 2\nu - 6\theta_1\kappa - 2\theta_2\kappa + 2\theta_3\kappa)P^2 \frac{\partial P}{\partial x} \end{aligned} \tag{9}$$

respectively.

Next, considering the traveling wave solution $P(x,t) = P(\tau)$ where $\tau = B(x - vt)$, where B and v are constants, eq. (8) becomes

$$
\begin{aligned}
&(\omega+\alpha\kappa+a\kappa^2)P \\
&+[\kappa(\lambda-\kappa\theta_1-\kappa\theta_2-\kappa\theta_3)-b]P^3 \\
&-aB^2P''+6PP'^2B^2\theta_1 \\
&+(3\theta_1+\theta_2+\theta_3)P^2B^2P''=0
\end{aligned} \tag{10}
$$

where prime denotes differentiation with respect to τ. The imaginary part leads to the relations

$$v=-\alpha-2a\kappa \tag{11}$$

and

$$3\lambda+2v-2\kappa(3\theta_1+\theta_2-\theta_3)=0 \tag{12}$$

Eq. (11) gives the speed of the soliton and eq. (12) is the constraint relation that must be valid in order for the solitons to exist.

Now, Eq. (10) can be written in the form

$$P''=A_1P+A_2P^3+A_3PP'^2+A_4P^2P'' \tag{13}$$

where,

$$A_1=\frac{\omega+\alpha\kappa+a\kappa^2}{aB^2} \tag{14-1}$$

$$A_2=\frac{\kappa(\lambda-\kappa\theta_1-\kappa\theta_2-\kappa\theta_3)-b}{aB^2} \tag{14-2}$$

$$A_3=\frac{6\theta_1}{a} \tag{14-3}$$

$$A_4=\frac{3\theta_1+\theta_2+\theta_3}{a} \tag{14-4}$$

Applying the mapping method, we can assume the solution structure of eq. (13) in the form [6, 7, 9]

$$P(\tau)=a_0+a_1f(\tau) \tag{15}$$

where f satisfies eq. (3). Substituting eq. (15) into eq. (13) and using eq. (3), we obtain a polynomial in f given by

$$
\begin{aligned}
&a_1pf+a_1qf^3=A_1(a_0+a_1f) \\
&+A_2(a_0^3+3a_0^2a_1f+3a_0a_1^2f^2+a_1^3f^3) \\
&+A_3\begin{pmatrix} a_0a_1^2\tau+a_1^3\tau f+a_0a_1^2pf^2 \\ +a_1^3pf^3+\frac{1}{2}a_0a_1^2qf^4 \end{pmatrix} \\
&+A_4\begin{bmatrix} a_0^2a_1pf+2a_0a_1^2pf^2 \\ +(a_0^2a_1q+a_1^3p)f^3 \\ +2a_0a_1^2qf^4+a_1^3qf^5 \end{bmatrix}
\end{aligned} \tag{16}
$$

Equating the coefficients of different powers of f in eq. (16), we arrive at the following algebraic equations:

$$
\begin{aligned}
&f^5:\frac{1}{2}a_1^3qA_3+a_1^3qA_4=0 \\
&\Rightarrow\frac{1}{2}A_3+A_4=0
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&f^4:\frac{1}{2}a_0a_1^2qA_3+2a_0a_1^2qA_4=0 \\
&\Rightarrow\frac{1}{2}A_3+2A_4=0
\end{aligned} \tag{18}
$$

Eqs. (17) and (18) lead us to $A_3=0$ and $A_4=0$. This gives rise to $\theta_1=0$ and $\theta_2=\theta_3=0$. From the coefficients of f^3, f^2 and the constant term, we obtain

$$a_0=0,\ a_1=\pm\sqrt{\frac{q}{A_2}},\ A_1=p \tag{19}$$

So, we can easily see that a_1 can be written as

$$a_1=\pm\sqrt{\frac{q(\omega+\alpha\kappa+a\kappa^2)}{p(\kappa\lambda-b)}} \tag{20}$$

Case-1: $p=-(1+m^2)$, $q=2m^2$, $\tau=1$

Here, eq. (3) gives $f(\tau)=\mathrm{sn}(\tau)$. In this case, eq. (4) gives rise to the periodic wave solution [8]

$$
\begin{aligned}
&q(x,t)=\pm\sqrt{\frac{2m^2(\omega+\alpha\kappa+a\kappa^2)}{(1+m^2)(b-\kappa\lambda)}} \\
&\times\mathrm{sn}[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)}
\end{aligned} \tag{21}
$$

As $m \to 1$, eq.(21) one recovers dark soliton solution

$$q(x,t) = \pm\sqrt{\frac{\omega + \alpha\kappa + a\kappa^2}{b - \kappa\lambda}} \times \tanh[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \quad (22)$$

Case-2: $p = 2m^2 - 1$, $q = -2m^2$, $\tau = 1 - m^2$

So, eq.(3) yields $f(\tau) = \mathrm{cn}(\tau)$. In this case, eq. (4) gives rise to the periodic wave solution [8]

$$q(x,t) = \pm\sqrt{\frac{2m^2(\omega + \alpha\kappa + a\kappa^2)}{(2m^2 - 1)(b - \kappa\lambda)}} \times \mathrm{cn}[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \quad (23)$$

As $m \to 1$, eq.(23) one obtains bright soliton solution

$$q(x,t) = \pm\sqrt{\frac{2(\omega + \alpha\kappa + a\kappa^2)}{b - \kappa\lambda}} \times \mathrm{sech}[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \quad (24)$$

Case-3: $p = -(1 + m^2)$, $q = 2$, $\tau = m^2$

Here, eq.(3) gives $f(\tau) = \mathrm{ns}(\tau)$. Therefore, eq. (4) gives rise to the periodic wave solution [8]

$$q(x,t) = \pm\sqrt{\frac{2(\omega + \alpha\kappa + a\kappa^2)}{(1 + m^2)(b - \kappa\lambda)}} \times \mathrm{ns}[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \quad (25)$$

As $m \to 1$, eq.(25) leads us to the singular soliton solution

$$q(x,t) = \pm\sqrt{\frac{\omega + \alpha\kappa + a\kappa^2}{b - \kappa\lambda}} \times \coth[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \quad (26)$$

These solitons and doubly periodic solutions, listed in (21)-(26) immediately introduce the constraint condition

$$(b - \kappa\lambda)(\omega + \alpha\kappa + a\kappa^2) > 0 \quad (27)$$

Thus, the solitons and doubly periodic functions will exist provided the constraint relation of the parameters hold.

3.2 Parabolic law nonlinearity

The equation under consideration, for this law of nonlinearity, is

$$\begin{aligned} &iq_t + aq_{xx} + \left(b_1|q|^2 + b_2|q|^4\right)q = i\alpha q_x \\ &+ i\lambda(|q|^2 q)_x + i\nu(|q|^2)_x q \\ &+ \theta_1(|q|^2 q)_{xx} + \theta_2|q|^2 q_{xx} + \theta_3 q^2 q^*_{xx} = 0 \end{aligned} \quad (28)$$

This law is commonly known as the cubic-quintic nonlinearity. The second term of nonlinearity on the left hand side of (28) is large for the case of p-toluene sulfonate crystals. This law arises in the nonlinear interaction between Langmuir waves and electrons. It describes the nonlinear interaction between the high frequency Langmuir waves and the ion-acoustic waves by pondermotive forces.

Substituting eq. (5) into eq. (28) and considering the traveling wave solution as in section 3, the imaginary part remains the same as before and the real part becomes

$$\begin{aligned} &P'' = A_1P + A_2P^3 + A_3PP'^2 \\ &+ A_4P^2P'' + A_5P^5 \end{aligned} \quad (29)$$

where, A_1, A_2, A_3 and A_4 are as in eq. (14) with b replaced by b_1 and $A_5 = -b_2/(aB^2)$.

Assuming the solution of eq. (29) in the form of eq. (15) and using eq. (3), we get a fifth degree polynomial in f. The coefficients of different powers of f give rise to a set of algebraic equations whose solutions give

$$a_0 = 0, \; a_1 = \sqrt{\frac{apB^2 - \omega - \alpha\kappa - a\kappa^2}{6B^2\tau\theta_1}} \quad (30)$$

and get the constraint condition

$$q\tau A_3^2 + 2q\tau A_3A_4 - 2A_1A_5 + 2pA_5 = 0 \quad (31)$$

Case-1: $p = -(1 + m^2)$, $q = 2m^2$, $\tau = 1$

Here, eq. (3) gives $f(\tau) = \mathrm{sn}(\tau)$. In this case, eq. (28) gives rise to the periodic wave solution [8]

$$q(x,t) = \pm\sqrt{-\frac{a(1 + m^2)B^2 + \omega + \alpha\kappa + a\kappa^2}{6B^2\theta_1}} \times \mathrm{sn}[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \quad (32)$$

As $m \to 1$, eq.(32) leads us dark soliton solution

$$q(x,t)=\pm\sqrt{-\frac{2aB^2+\omega+\alpha\kappa+a\kappa^2}{6B^2\theta_1}}\times\tanh[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)} \quad (33)$$

Case-2: $p=2m^2-1$, $q=-2m^2$, $\tau=1-m^2$

So, eq.(3) yields $f(\tau)=\mathrm{cn}(\tau)$. In this case, eq. (28) gives rise to the periodic wave solution [8]

$$q(x,t)=\pm\sqrt{\frac{a(2m^2-1)B^2-\omega-\alpha\kappa-a\kappa^2}{6B^2(1-m^2)\theta_1}}\times\mathrm{cn}[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)} \quad (34)$$

As $m\to 1$, eq.(34) does not give rise to a solitary wave solution.

Case-3: $p=-(1+m^2)$, $q=2$, $\tau=m^2$

Here, eq.(3) gives $f(\tau)=\mathrm{ns}(\tau)$. In this case, eq. (28) gives rise to the periodic wave solution [8]

$$q(x,t)=\pm\sqrt{-\frac{a(1+m^2)B^2+\omega+\alpha\kappa+a\kappa^2}{6B^2m^2\theta_1}}\times\mathrm{ns}[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)} \quad (35)$$

As $m\to 1$, eq.(35) leads us to the singular solitary wave solution

$$q(x,t)=\pm\sqrt{-\frac{2aB^2+\omega+\alpha\kappa+a\kappa^2}{6B^2\theta_1}}\times\coth[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)} \quad (36)$$

It needs to be noted that the doubly periodic functions, for this law of nonlinearity, given by (32) and (35) will exist provided

$$\theta_1\left\{a(1+m^2)B^2+\omega+\alpha\kappa+a\kappa^2\right\}<0 \quad (37)$$

Consequently, dark soliton and singular soliton will exist if

$$\theta_1\left(2aB^2+\omega+\alpha\kappa+a\kappa^2\right)<0 \quad (38)$$

Finally, the cnoidal wave solution given by (34) will exist with the constraint

$$\theta_1\left\{a(2m^2-1)B^2-\omega-\alpha\kappa+a\kappa^2\right\}>0 \quad (39)$$

4. Conclusion

This paper retrieved soliton solutions to the NLSE in optical metamaterials with Kerr and parabolic law nonlinearity. The mapping method is applied to obtain these solutions. The results of this paper came with certain constraints that must hold for these solitons to exist. These soliton solutions are recovered after a limiting process applied to doubly periodic functions when the modulus of ellipticity approached unity. This approach is therefore a very unique method to derive soliton solutions.

Later the results will be extended to the case when several perturbation terms will be considered. Better yet, soliton perturbation theory will be applied to give the adiabatic variation of these soliton parameters. Several other integration tools will be adopted to obtain soliton and other solutions. The results of those researches are awaited at this time.

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