Solitons in optical metamaterials with parabolic law nonlinearity and spatio-temporal dispersion

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This work studies the optical solitons in optical nonlinear metamaterials. The parabolic law nonlinearity, spatio-temporal dispersion, inter-modal dispersion, self-steepening as well as nonlinear dispersion are taken into account. Two integration approaches that are Ricatti equation expansion method and ansatz scheme are adopted to obtain optical solitons. Finally, analytical dark soliton, bright soliton, singular soliton as well as singular periodic solutions are reported. The corresponding constraint conditions in order for these exact solutions to exist are derived.

1. Introduction

Metamaterials (MMs) are a new type of artificial synthetic material and have lots of fascinating electromagnetic properties that the traditional optical materials in nature have not [1-7]. The controllable dielectric permittivity and magnetic permeability make MMs with adjustable linear and nonlinear parameters. These unique properties provide a way to realize the optical solitons control in experiments.

Recently, the study of optical solitons in MMs has attracted many researchers attention. Xiang et al. [4] studied the controllable Raman soliton self-frequency shift in MMs. Saha and Sarma [5] studied the modulation instability in MMs with cubic–quintic nonlinearities. In our previous work [6, 7], we investigated the MMs with Kerr nonlinearity, and constructed the explicit bright, dark as well as singular solitons by employing the ansatz approach and simplest equation method. However, to the best of our knowledge, no studies on the optical solitons in MMs with parabolic law nonlinearity and spatio-temporal dispersion are reported thus far. This work therefore fills in the gap.

The nonlinear dynamical model that describes the propagation of pulses in nonlinear MMs is given by the nonlinear Schrödinger equation (NLSE). In the presence of parabolic law nonlinearity, spatio-temporal dispersion (STD), inter-modal dispersion (IMD), self-steepening (SS) as well as nonlinear dispersion (ND), the governing equation reads

$$iq_t + aq_{xx} + b|q|^2 q + c|q|^4 q + dq_{xt} + i\lambda q_x + is(|q|^2 q)_x + i\mu(|q|^2)_x q + \theta_1(|q|^2 q)_{xx} + \theta_2|q|^2 q_{xx} + \theta_3 q^2 q_{xx}^* = 0$$

(1)

In Equation (1), the unknown variable $q(x,t)$ represents the wave profile, while x and t are the space and time variables respectively. The first and second terms are the linear temporal evolution term and group velocity dispersion (GVD), while third and fourth terms account for the parabolic law nonlinearity, and fifth, sixth, seventh and eighth terms represent the STD, IMD, SS and ND respectively. Finally, the last three terms with θ_l for $l = 1, 2, 3$ appear in the context of metamaterials [6, 7].

It should be noted that the STD is considered in this research. Previous studies have shown that the nonlinear dynamical model is ill-posed if we fail to consider the STD [8,9]. In addition, the parabolic law nonlinearity, originates from the nonlinear interaction between Langmuir waves and electrons, is taken into consideration in this work. For a parabolic law nonlinear medium [10-18], the expression of refractive index is given by $n = n_0 + (3\chi^{(3)}/8n_0)I + (5\chi^{(5)}/16n_0)I^2$ [10-12], in which n_0, $\chi^{(3)}$, and $\chi^{(5)}$ are linear refractive index of

medium, third-order susceptibility and fifth-order susceptibility respectively.

This work focus on Equation (1), which will be integrated by the aid of Ricatti equation expansion method and ansatz scheme. We report analytical dark soliton, bright soliton, singular soliton as well as singular periodic solutions to Equation (1). The presented results may have an important role in the understanding optical solitons propagation in optical nonlinear metamaterials.

2. Mathematical analysis

First we introduce the hypothesis [6,7,19]

$$q(x,t) = P[\eta(x,t)] \exp[i\phi(x,t)] \qquad (2)$$

where $P[\eta(x,t)]$ is the amplitude component of solitons, and $\eta(x,t) = B(x - vt)$, in which B and v represent inverse width and velocity of soliton. Additionally, $\phi(x,t)$ is the phase component of solitons and is defined as $\phi(x,t) = -\kappa x + \omega t + \theta$, in which κ, ω, and θ represent soliton frequency, wave number and phase constant respectively.

Putting Equation (2) into Equation (1) and then decomposing into real and imaginary parts yields a pair of relations. The real part gives

$$(a - dv)B^2 P'' + 6\theta_1 B^2 P(P')^2 + (3\theta_1 + \theta_2 + \theta_3)B^2 P^2 P''$$
$$- (\omega + a\kappa^2 - d\omega\kappa - \lambda\kappa)P$$
$$+ (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa)P^3 + cP^5 = 0$$
$$(3)$$

and the imaginary part gives

$$(\kappa d - 1)v - 2a\kappa + d\omega + \lambda$$
$$+ (3s + 2\mu - 6\theta_1\kappa - 2\theta_2\kappa + 2\theta_3\kappa)P^2 = 0 \qquad (4)$$

Using the homogeneous balance principle to the imaginary part (4) leads to

$$v = \frac{2a\kappa - d\omega - \lambda}{\kappa d - 1} \qquad (5)$$

$$3s + 2\mu - 6\theta_1\kappa - 2\theta_2\kappa + 2\theta_3\kappa = 0 \qquad (6)$$

Equation (5) gives the soliton velocity, while Equation (6) gives the first constrains condition for the existence of solitons.

Then the main task of this paper is study the real part (3). It needs to be noted that a special case for Equation (3) with $\theta_1 = 0$, $\theta_2 + \theta_3 = 0$, and $c = 0$ was investigated analytically in 2014 [7]. This work is an extension and generalization of these earlier results. In the following subsections, two integration tools that

are Ricatti equation expansion method and ansatz scheme will be adopted to derive solitons in details.

2.1 Ricatti equation expansion method

In this subsection, we will perform the Ricatti equation expansion method to Equation (3).

Suppose that $P(\eta)$ satisfies the Ricatti equation in the form [7,13-15]

$$P'(\eta) = M + NP^2(\eta) \qquad (7)$$

where M and N are non-zero real constants. All exact solutions to Equation (7) are listed in Refs. [7, 19-21].

Inserting Equations (7) into Equation (3) leads to

$$\left\{ 2(a - dv)B^2 MN + 6\theta_1 B^2 M^2 - (\omega + a\kappa^2 - d\omega\kappa - \lambda\kappa) \right\}P$$
$$+ \left\{ \begin{array}{l} 2(a - dv)B^2 N^2 + 12\theta_1 B^2 MN + 2(3\theta_1 + \theta_2 + \theta_3)B^2 MN \\ + (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa) \end{array} \right\}P^3$$
$$+ \left\{ 6\theta_1 N^2 B^2 + 2(3\theta_1 + \theta_2 + \theta_3)B^2 N^2 + c \right\}P^5 = 0$$
$$(8)$$

Setting the coefficients of $P^n(\eta)$ for $n = 1, 3, 5$ to zero leads to

$$2(a - dv)B^2 MN + 6\theta_1 B^2 M^2 - (\omega + a\kappa^2 - d\omega\kappa - \lambda\kappa) = 0 \quad (9)$$

$$2(a - dv)B^2 N^2 + 12\theta_1 B^2 MN + 2(3\theta_1 + \theta_2 + \theta_3)B^2 MN$$
$$+ (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa) = 0 \qquad (10)$$

$$6\theta_1 N^2 B^2 + 2(3\theta_1 + \theta_2 + \theta_3)B^2 N^2 + c = 0 \qquad (11)$$

Solving these algebraic equations (9)-(11) yields

$$\omega = \frac{(dv - a)cMN - 3\theta_1 cM^2 + (\lambda - a\kappa)(6\theta_1 + \theta_2 + \theta_3)\kappa N^2}{(1 - d\kappa)(6\theta_1 + \theta_2 + \theta_3)N^2} \quad (12)$$

$$B = \sqrt{-\frac{c}{2(6\theta_1 + \theta_2 + \theta_3)N^2}} \qquad (13)$$

$$(dv - a)cN - (9\theta_1 + \theta_2 + \theta_3)cM$$
$$+ (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa)(6\theta_1 + \theta_2 + \theta_3)N = 0 \qquad (14)$$

Equation (12) gives the wave number, while Equation (13) gives inverse width of soliton and Equation (14) gives the second constrain condition for the existence of solitons.

Additionally, Equation (13) introduces the third constraint condition that is given by

$$(6\theta_1 + \theta_2 + \theta_3)c < 0 \qquad (15)$$

Hence, finally analytical solutions to Equation (1) are obtained, which are listed as follows:

2.1.1 Singular periodic solutions

When $MN > 0$, Equation (1) admits the singular periodic solutions

$$P(\eta) = \frac{\sqrt{MN}}{N} \tan[\sqrt{MN} B(x - vt)] \qquad (16)$$
$$\times \exp[i(-\kappa x + \omega t + \theta)]$$

$$P(\eta) = -\frac{\sqrt{MN}}{N} \cot[\sqrt{MN} B(x - vt)] \qquad (17)$$
$$\times \exp[i(-\kappa x + \omega t + \theta)]$$

2.1.2 Soliton solutions

When $MN < 0$, Equation (1) admits the dark soliton

$$P(\eta) = -\frac{\sqrt{-MN}}{N} \tanh[\sqrt{-MN} B(x - vt)] \qquad (18)$$
$$\times \exp[i(-\kappa x + \omega t + \theta)]$$

and singular soliton

$$P(\eta) = -\frac{\sqrt{-MN}}{N} \coth[\sqrt{-MN} B(x - vt)] \qquad (19)$$
$$\times \exp[i(-\kappa x + \omega t + \theta)]$$

2.2. Ansatz scheme

In this subsection, we will perform the ansatz scheme [19,22-29] to obtain the dark, bright as well as singular solitons.

2.2.1 Dark soliton

For dark soliton, the starting hypothesis is given by

$$P(\eta) = l_0 + l_1 \tanh^n \eta \qquad (20)$$

where l_0 and l_1 are the real constants to be determined later.

Substituting the hypothesis (20) into Equation (3) leads to

$$(a - dv)l_1 B^2[n(n-1)\tanh^{n-2}\eta - 2n^2 \tanh^n \eta$$
$$+ n(n+1)\tanh^{n+2}\eta]$$
$$+ 6\theta_1 B^2 l_1^2 (l_0 + l_1 \tanh^n \eta)[n \tanh^{n-1}\eta - n \tanh^{n+1}\eta]^2$$
$$+ (3\theta_1 + \theta_2 + \theta_3)l_1 B^2 (l_0 + l_1 \tanh^n \eta)^2 [n(n-1)\tanh^{n-2}\eta$$
$$- 2n^2 \tanh^n \eta + n(n+1)\tanh^{n+2}\eta]$$
$$- (\omega + a\kappa^2 - d\omega\kappa - \lambda\kappa)(l_0 + l_1 \tanh^n \eta)$$
$$+ (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa)(l_0 + l_1 \tanh^n \eta)^3$$
$$+ c(l_0 + l_1 \tanh^n \eta)^5 = 0$$
$$(21)$$

Using the homogeneous balance method to Equation (21) leads to

$$n = 1 \qquad (22)$$

Then, upon setting the coefficients of each power of $\tanh(\eta)$ to zero yields a set of algebraic equations for the parameters. Solving those equations gives

$$l_0 = 0 \qquad (23)$$

$$\omega = \frac{(a - dv)cl_1^2 - 3\theta_1 cl_1^4 + (\lambda - a\kappa)(6\theta_1 + \theta_2 + \theta_3)\kappa}{(6\theta_1 + \theta_2 + \theta_3)(1 - d\kappa)} \qquad (24)$$

$$B = \sqrt{-\frac{cl_1^2}{2(6\theta_1 + \theta_2 + \theta_3)}} \qquad (25)$$

$$(a - dv)c - (9\theta_1 + \theta_2 + \theta_3)cl_1^2 \qquad (26)$$
$$- (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa)(6\theta_1 + \theta_2 + \theta_3) = 0$$

where l_1 is an arbitrary non-zero real constant.

Equation (24) gives the wave number, while Equation (25) gives inverse width of soliton and Equation (26) gives the constrain condition for the existence of solitons. Additionally, Equation (25) posses a constrain condition that is given by Equation (15).

Hence, finally analytical dark soliton to Equation (1) is derived that is given by

$$P(\eta) = l_1 \tanh[B(x - vt)] \exp[i(-\kappa x + \omega t + \theta)] \qquad (27)$$

2.2.2. Bright soliton

For singular soliton, the starting hypothesis is given by

$$P(\eta) = l_0 + l_1 \mathrm{sech}^n \eta \qquad (28)$$

Substituting the hypothesis (28) into Equation (3) leads to

$$(a - dv)B^2[n^2 l_1 \mathrm{sech}^n \eta - n(n+1)l_1 \mathrm{sech}^{n+2}\eta]$$
$$+ 6\theta_1 B^2 (l_0 + l_1 \mathrm{sech}^n \eta)l_1^2 n^2 \mathrm{sech}^{2n}\eta(1 - \mathrm{sech}^2\eta)$$
$$+ (3\theta_1 + \theta_2 + \theta_3)B^2 (l_0 + l_1 \mathrm{sech}^n \eta)^2$$
$$\times [n^2 l_1 \mathrm{sech}^n \eta - n(n+1)l_1 \mathrm{sech}^{n+2}\eta]$$
$$- (\omega + a\kappa^2 - d\omega\kappa - \lambda\kappa)(l_0 + l_1 \mathrm{sech}^n \eta)$$
$$+ (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa)(l_0 + l_1 \mathrm{sech}^n \eta)^3$$
$$+ c(l_0 + l_1 \mathrm{sech}^n \eta)^5 = 0$$
$$(29)$$

Balancing principle gives

$$n = 1 \qquad (30)$$

Then, upon setting the coefficients of each power of $\mathrm{sech}(\eta)$ to zero yields a set of algebraic equations involving parameters. Solving those equations gives

$$\omega = \frac{(a-dv)cl_1^2 + 2(6\theta_1 + \theta_2 + \theta_3)(\lambda - a\kappa)\kappa}{2(6\theta_1 + \theta_2 + \theta_3)(1 - d\kappa)} \quad (31)$$

$$B = \sqrt{\frac{cl_1^2}{2(6\theta_1 + \theta_2 + \theta_3)}} \quad (32)$$

$$\begin{aligned}
&2(dv-a)c + (9\theta_1 + \theta_2 + \theta_3)cl_1^2 + 2(6\theta_1 + \theta_2 + \theta_3) \\
&\times (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa) = 0
\end{aligned} \quad (33)$$

where l_0 is given by Equation (23), and l_1 is an arbitrary non-zero real constant.

Equation (31) gives the wave number, while Equation (32) gives inverse width of soliton and Equation (33) gives the constrain condition for the existence of solitons. Additionally, Equation (32) posses an other constrain condition that is given by

$$(6\theta_1 + \theta_2 + \theta_3)c > 0 \quad (34)$$

Hence, finally analytical dark soliton to Equation (1) is derived that is given by

$$P(\eta) = l_1 \mathrm{sech}[B(x - vt)]\exp[i(-\kappa x + \omega t + \theta)] \quad (35)$$

2.2.3. Singular soliton

For singular soliton, the starting hypothesis is given by

$$P(\eta) = l_0 + l_1 \coth^n \eta \quad (36)$$

Substituting the hypothesis (36) into Equation (3) leads to

$$\begin{aligned}
&(a-dv)B^2 l_1[n(n-1)\coth^{n-2}\eta(\coth^2\eta - 1)^2 \\
&+ 2n\coth^n\eta(\coth^2\eta - 1)] \\
&+ 6\theta_1 B^2 l_1^2(l_0 + l_1\coth^n\eta)[-n\coth^{n-1}\eta(\coth^2\eta - 1)]^2 \\
&+ (3\theta_1 + \theta_2 + \theta_3)B^2 l_1(l_0 + l_1\coth^n\eta)^2 \\
&\times [n(n-1)\coth^{n-2}\eta(\coth^2\eta - 1)^2 + 2n\coth^n\eta(\coth^2\eta - 1)] \\
&- (\omega + a\kappa^2 - d\omega\kappa - \lambda\kappa)(l_0 + l_1\coth^n\eta) \\
&+ (b - \theta_1\kappa^2 - \theta_2\kappa^2 - \theta_3\kappa^2 + s\kappa)(l_0 + l_1\coth^n\eta)^3 \\
&+ c(l_0 + l_1\coth^n\eta)^5 = 0
\end{aligned}$$
$$(37)$$

Balancing principle gives

$$n = 1 \quad (38)$$

Then, upon setting the coefficients of each power of $\coth(\eta)$ to zero yields a set of algebraic equation for

the parameters. Solving those equations, we obtain the values of l_0, ω, and B that are same as Equations (23)-(25). In addition, the constrains given by Equations (26) and (15), for the solitons to exist.

Hence, finally analytical singular soliton to Equation (1) is derived that is given by

$$P(\eta) = l_1 \coth[B(x - vt)]\exp[i(-\kappa x + \omega t + \theta)] \quad (39)$$

3. Conclusion

The NLSE (1), which can be used to describe the propagation of optical solitons in MMs, has been investigated analytically in this paper. The perturbations, for parabolic law nonlinearity, are STD, IMD, SS and ND. There are two integration approaches that are implemented. They are the Ricatti equation expansion method and ansatz scheme. The first approach retrieved analytical singular periodic solutions, dark as well as singular solitons. The second algorithm retrieved exact dark, bright as well as singular solitons. These results are useful in describing optical solitons propagation in optical nonlinear MMs.

References

[1] A. A. Zharov, I. V. Shadrivov, Y. S. Kivshar, Phys. Rev. Lett. **91**, 037401 (2003).

[2] R. G. Li, R. C. Yang, Z. Y. Xu, Phys. Rev. E **82**, 046603 (2010).

[3] S. Wen, Y. Wang, W. Su, Y. Xiang, X. Fu, D. Fan, Phys. Rev. E **73**, 036617 (2006).

[4] Y. Xiang, X. Dai, S. Wen, J. Guo, D. Fan, Phys. Rev. A **84**, 033815 (2011)

[5] M. Saha, A. K. Sarma, Opt. Commun. **291**, 321, (2013).

[6] A. Biswas, K. R. Khan, M. F. Mahmood, M. Belic, Optik, **125**, 3299, (2014).

[7] A. Biswas, M. Mirzazadeh, M. Savescu, D. Milovic, K. R. Khan, M. F. Mahmood, M. Belic, J. Mod. Opt. 2014.

[8] M. Savescu, E. M. Hilal, A. A. Alshaery, A. H. Bhrawy, L. Moraru, A. Biswas, J. Optoelectron. Adv. Mater. **16**, 619, (2014).

[9] M. Savescu, A. H. Bhrawy, A. A. Alshaery, E. M. Hilal, K. R. Khan, M. F. Mahmood, A. Biswas, J. Mod. Opt. **61**, 441, (2014).

[10] Q. Zhou, J. Mod. Opt. **61**, 500, (2014).

[11] Q. Zhou, D. Z. Yao, X. N. Liu, F. Chen, S. J. Ding,

Y. F. Zhang, F. Chen, Opt. Laser Technol. **51**, 32, (2013).

[12] Q. Zhou, D. Z. Yao, F. Chen, J. Mod. Opt. **60**, 1652, (2013).

[13] Q. Zhou, Q. Zhu, A.H. Bhrawy, L. Moraru, A. Biswas, Optoelectron. Adv. Mat. **8**, 800 (2014).

[14] Q. Zhou, D. Z. Yao, Z. H. Cui, J. Mod. Opt. **59**, 57 (2012).

[15] H. Triki, F. Azzouzi, P. Grelu, Opt. Commun. **309**, 71 (2013).

[16] A. Biswas, A. Yildirim, T. Hayat, O. M. Aldossary, R. Sassaman, Proc. Romanian Acad. A **13**, 32 (2012).

[17] B. Ahmed, A. Biswas, Proc. Romanian Acad. A **14**, 111 (2013).

[18] A. Biswas, K. R. Khan, A. Rahman, A. Yildirim, T. Hayat, O. M. Aldossary, J. Optoelectron. Adv. Mater. **14**, 571 (2012).

[19] Q. Zhou, Q. Zhu, A. Biswas, Opt. Appl. Forthcoming 2014. accepted.

[20] N. K. Vitanov, Commun. Nonlinear Sci. Numer. Simul. **15**, 2050 (2010).

[21] Q. Zhou, Optik, **125**, 5432 (2014).

[22] Q. Zhou, Q, P. Zhu, Waves Random Complex Media, 2014. doi: 10.1080/17455030.2014.956847.

[23] Q. Zhou, Q, P. Zhu, M. Savescu, A. H. Bhrawy, A. Biswas, Proc. Romanian Acad. A, 2014. accepted.

[24] A. K. Sarma, A. Biswas, Opt. Appl. 44, 205 (2014).

[25] A. H. Bhrawy, A. A. Alshaery, E. M. Hilal, Wayne N. Manrakhan, M. Savescu, A. Biswas, J. Nonlinear Opt. Phys, Mat. **23**, 1450014 (2014).

[26] A. Biswas, A. H. Bhrawy, A. A. Alshaery, E. M. Hilal, Optik, **125**, 4932 (2014).

[27] A. Biswas, M. Mirzazadeh, Optik, **125**, 4603 (2014).

[28] R. Morris, P. Masemola, A.H. Kara, A. Biswas, Optik, **124,** 5116 (2013).

[29] S. Shwetanshumala, N. Asif, S. Konar, A. Biswas, Optik, **124,** 229 (2013).

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